Integration

Study Development Worksheet

Example

Evaluate the following integral:

$$\int_0^{\frac{\pi}{2}} cos(x)\,\mathrm{d}x$$

Answer

Using the rules for integrating trigonometric functions:

$$\int_0^{\frac{\pi}{2}} cos(x)\,\mathrm{d}x = [sin(x)]_0^{\frac{\pi}{2}}$$

Then, evaluating the integral between the limits:

$$[\sin(x)]_0^{\frac{\pi}{2}} = sin\left(\frac{\pi}{2}\right) - sin(0) = 1 - 0 = 1$$

Questions

Calculate the following:

1. $\int x^3 + 2x^2 + 4\,\mathrm{d}x$
2. $\int x^{\frac{1}{2}}\,\mathrm{d}x$
3. $\int e^{2x} + cos(2x)\,\mathrm{d}x$
4. $\int_2^5 3x^3 + 4\,\mathrm{d}x$
5. $\int_0^1 (4x+6)e^{x^2+3x}\,\mathrm{d}x$
6. $\int -tan(x)\,\mathrm{d}x$
7. $\int (3x^2 - 52x + 147)\,sin((x-7)^3)\,\mathrm{d}x$
8. $\int_0^{\frac{\pi}{2}} 5\,sin^2(2x)\,cos(2x)\,\mathrm{d}x$
9. $\int_{\pi}^{\frac{3\pi}{2}} x\,sin(x)\,\mathrm{d}x$
10. $\int x\,log(x)\,\mathrm{d}x$
11. $\int_0^{\frac{\pi}{2}} e^x\,cos(x)\,\mathrm{d}x$

Answers

1. $\int x^3 + 2x^2 + 4 \, \mathrm{d}x = \frac{x^4}{4} + \frac{2x^3}{3} + 4x + c$

2. $\int x^{\frac{1}{2}} \, \mathrm{d}x = \frac{2x^{\frac{3}{2}}}{3} + c$

3. $\int e^{2x} + cos(2x) \, \mathrm{d}x = \frac{e^{2x}}{2} + \frac{sin(2x)}{2} + c$

4. $\int_2^5 3x^3 + 4\mathrm{d}x = \left[\frac{3x^4}{4} + 4x\right]_2^5$

   $= \frac{3(5^4)}{4} + 4(5) - \left(\frac{3(2)^4}{4} + 4(2)\right)$

   $= 468.75$

5. $\frac{\mathrm{d}(x^2+3x)}{\mathrm{d}x} = 2x + 3$

   Therefore, $\frac{\mathrm{d}e^{x^2+3x}}{\mathrm{d}x} = (2x+3)e^{x^2+3x}$.

   $\int_0^1 (4x+6)e^{x^2+3x} \, \mathrm{d}x = \int_0^1 2(2x+3)e^{x^2+3x} \, \mathrm{d}x = \left[2e^{x^2+3x}\right]_0^1$

   $= 2e^{1^2+3(1)} - 2e^{0^2-3(0)} = 2e^4 - 2e^0$

   $= 2e^4 - 2 \; (= 107.1963)$

6. Begin by rewriting the integral in the format

$$\int -tan(x) \, \mathrm{d}x = \int \frac{-sin(x)}{cos(x)} \mathrm{d}x$$

   Since $-sin(x) = \frac{\mathrm{d}\,cos(x)}{\mathrm{d}x}$, the integral is evaluated as

$$\int \frac{-sin(x)}{cos(x)} \mathrm{d}x = log(cos(x)) + c$$

7. Set $u = (x-7)^3$, which gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3(x-7)^2$$

Therefore, $\mathrm{d}x = \frac{\mathrm{d}u}{3(x-7)^2}$, and write

$$\int (3x^2 - 52x + 147)\, sin((x-7)^3)\, \mathrm{d}x = \int (3x^2 - 52x + 147)\, sin(u) \frac{\mathrm{d}u}{3(x-7)^2}$$

This simplifies to give

$$\int 3(x-7)^2\, sin(u) \frac{\mathrm{d}u}{3(x-7)^2}$$

And then, $\int sin(u)\, \mathrm{d}u = -cos(u) + c$.

Finally, substitute $u = (x-7)^3$ back in:

$$\int (3x^2 - 52x + 147)\, sin((x-7)^3)\, \mathrm{d}x = -cos((x-7)^3) + c$$

8. Begin by evaluating $\int 5\, sin^2(2x)\, cos(2x)\, \mathrm{d}x$:

Set $u = sin(2x)$. Therefore, $\frac{\mathrm{d}u}{\mathrm{d}x} = 2\, cos(2x)$, and so $\mathrm{d}x = \frac{\mathrm{d}u}{2\, cos(2x)}$.

Now, $\int 5\, sin^2(2x)\, cos(2x)\, \mathrm{d}x = \int 5u^2\, cos(2x) \frac{\mathrm{d}u}{2\, cos(2x)}$, which simplifies to give

$$\int \frac{5}{2} u^2 \mathrm{d}u$$

This gives $\int \frac{5}{2} u^2\, \mathrm{d}u = \frac{5u^3}{6} + c$.

u is then substituted back in:

$$\int 5\, sin^2(2x)\, cos(2x)\, \mathrm{d}x = \frac{5\, sin^3(2x)}{6} + c$$

Now, evaluate $\int_0^{\frac{\pi}{2}} 5\, sin^2(2x)\, cos(2x)\, \mathrm{d}x = \left[\frac{5\, sin^3(2x)}{6}\right]_0^{\frac{\pi}{2}}$.

This gives

$$\frac{5\, sin^3(\pi)}{6} - \frac{5\, sin^3(0)}{6} = 0 - 0 = 0$$

9. Using integration by parts, set $u = x$ and $v' = sin(x)$. Find $u' = 1$ and $v = -cos(x)$. Plug these values into the integration by parts formula to get

$$\int x\, sin(x)\, \mathrm{d}x = -x\, cos(x) - \int -cos(x)\, \mathrm{d}x$$

Evaluating this gives

$$\int x\, sin(x)\, \mathrm{d}x = -x\, cos(x) + sin(x) + c$$

Then, use this to evaluate

$$\int_{\pi}^{\frac{3\pi}{2}} x\, sin(x)\, \mathrm{d}x = [-x\, cos(x) + sin(x)]_{\pi}^{\frac{3\pi}{2}}$$

$$= -\frac{3\pi\, cos\left(\frac{3\pi}{2}\right)}{2} + \sin\left(\frac{3\pi}{2}\right) - (-\pi\, cos(\pi) + sin(\pi))$$

$$= -1 - \big(-\pi(-1)\big) = -1 - \pi$$

10. Using integration by parts, set $u = log(x)$ and $v' = x$. Find $u' = \frac{1}{x}$ and $v = \frac{x^2}{2}$. Then, plug these values into the integration by parts formula to get

$$\int x\, log(x)\, \mathrm{d}x = \frac{log(x)\, x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2} \mathrm{d}x$$

Simplify:

$$\int x\, log(x)\, \mathrm{d}x = \frac{log(x)\, x^2}{2} - \int \frac{x}{2} \mathrm{d}x$$

Calculating this gives:

$$\int x\, log(x)\, \mathrm{d}x = \frac{log(x)\, x^2}{2} - \frac{x^2}{4} + c$$

Therefore,

$$\int x\, log(x)\, \mathrm{d}x = \frac{x^2}{2}\left(log(x) - \frac{1}{2}\right) + c$$

11. Using integration by parts, set $u = e^x$ and $v' = cos(x)$. Find $u' = e^x$ and $v = sin(x)$.

Then, plug these values into the integration by parts formula to get

$$\int e^x\, cos(x)\, \mathrm{d}x = e^x\, sin(x) - \int e^x\, sin(x)\, \mathrm{d}x$$

Then, evaluate the new integral using integration by parts, to get

$$u = e^x, v' = sin(x)$$

$$u' = e^x, v = -cos(x)$$

$$\int e^x\, sin(x)\, \mathrm{d}x = -e^x\, cos(x) + c - \int e^x(-cos(x))\mathrm{d}x$$

Simplify this to get

$$\int e^x\, sin(x)\, \mathrm{d}x = -e^x\, cos(x) + c + \int e^x\, cos(x)\, \mathrm{d}x$$

Plug this back into the original integral to get

$$\int e^x\, cos(x)\, \mathrm{d}x = e^x\, sin(x) + c - \left(-e^x\, cos(x) + \int e^x\, cos(x)\, \mathrm{d}x\right)$$

Simplify:

$$\int e^x\, cos(x)\, \mathrm{d}x = e^x\, sin(x) + e^x\, cos(x) + c - \int e^x\, cos(x)\, \mathrm{d}x$$

Next, add $\int e^x\, cos(x)\, \mathrm{d}x$ to both sides:

$$2\int e^x\, cos(x)\, \mathrm{d}x = e^x(sin(x) + cos(x)) + c$$

Then divide both sides by 2:

$$\int e^x\, cos(x)\, \mathrm{d}x = \frac{1}{2}e^x(sin(x) + cos(x)) + c$$

Now, find

$$\int_0^{\frac{\pi}{2}} e^x \, cos(x)\, \mathrm{d}x = \left[\frac{1}{2}e^x(sin(x) + cos(x))\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}e^{\frac{\pi}{2}}\left(sin\left(\frac{\pi}{2}\right) + cos\left(\frac{\pi}{2}\right)\right) - \left(\frac{1}{2}e^0(sin(0) + cos(0))\right)$$

$$= \frac{1}{2}\left(e^{\frac{\pi}{2}}(1+0) - (0+1)\right)$$

$$= \frac{e^{\frac{\pi}{2}} - 1}{2} (= 1.90524)$$

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