# Eigenvectors, eigenvalues and diagonalisation

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## Eigenvectors and eigenvalues

The eigenvectors and eigenvalues of a matrix are the solutions to the equation

$$Ax = \lambda x$$

Where $A$ is a square matrix, $x$ is an eigenvector of $A$ and $\lambda$ is an eigenvalue of $A$.

We find eigenvectors and eigenvalues for a few reasons. When we are describing vast amounts of information, describing a matrix by its eigenvectors and values can reduce the amount of information we need to give. If $A$ is a transformation vector, we know that it will not affect $x$ other than to multiply by a scalar $\lambda$. The final section in this factsheet is about diagonalization and decomposition, which is a great use of eigenvectors and eigenvalues that make difficult calculations much easier.

We might look at this equation and see that $x = 0$ is an obvious solution, no matter what $A$ and $\lambda$ are. Unfortunately, one of the properties of an eigenvector is that it cannot be the zero vector, so we must look for other solutions.

(For further motivation to learn to find eigenvectors and eigenvalues, there are many listed here: Eigenvectors and Eigenvalues Wikipedia page )

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## Characteristic equation

The first step in finding the eigenvalues of a matrix is to find its characteristic equation:

$$det(A - \lambda I) = 0$$

This comes from the equation linking the matrix and its eigenvectors and values:

$$Ax = \lambda x$$

We rearrange this to get

$$Ax - \lambda x = 0$$

We rewrite $\lambda x = \lambda I\, x$, since $Ix = x$ for all $x$:

$$Ax - \lambda Ix = 0$$

(If $A$ is an nxn matrix, we choose $I_n$ as $I$, where $I_n$ is the nxn identity matrix.)

We then factorise:

$$(A - \lambda I)x = 0$$

**Note:** this is why we replace $x$ with $Ix$, since a matrix minus a scalar isn't something we can calculate.

We want to find a non-zero value for $x$. If $(A - \lambda I)$ has an inverse, $(A - \lambda I)^{-1}$, then we find that

$$(A - \lambda I)^{-1}(A - \lambda I)x = (A - \lambda I)^{-1}0$$

Which gives

$$Ix = (A - \lambda I)^{-1}0$$

And so

$$x = 0$$

Therefore, we know $(A - \lambda I)$ cannot be invertible, and so $det(A - \lambda I) = 0$ (by the property that a matrix is invertible if and only if its determinant is equal to 0).

So, we have that $det(A - \lambda I) = 0$, which we can solve to find values for $\lambda$.

For example, for the matrix $A = \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}$, we find the characteristic equation by:

$$det\left(\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$det\left(\begin{pmatrix} -5-\lambda & -6 \\ 3 & 4-\lambda \end{pmatrix}\right) = 0$$

$$(-5-\lambda)(4-\lambda) - (-6 \times 3) = 0$$

$$-20 + 5\lambda - 4\lambda + \lambda^2 + 18 = 0$$

Which we simplify to find

$$\lambda^2 + \lambda - 2 = 0$$

## Eigenvalues

We find the eigenvalues of $A$ by solving the characteristic equation.

For our earlier example, we have a characteristic equation of

$$\lambda^2 + \lambda - 2 = 0$$

We factorise to find

$$(\lambda - 1)(\lambda + 2) = 0$$

So, our eigenvalues for $A = \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}$ are $\lambda = 1$ and $\lambda = -2$.

If you'd like, you can label these as $\lambda_1 = 1$ and $\lambda_2 = -2$, just to make it easier to differentiate between them.

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## Eigenvectors

We find the eigenvectors by solving the equation

$$Ax = \lambda x$$

For each of the values of $\lambda$ we have just found.

For example:

$\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} x = 1x$, and $\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} x = -2x$.

We know that $x$ must be a 2x1 vector, or the calculation $Ax$ cannot be carried out, so we write $x = \begin{pmatrix} a \\ b \end{pmatrix}$ and find values for $a$ and $b$ in each case:

For $\lambda_1 = 1$: $\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1 \begin{pmatrix} a \\ b \end{pmatrix}$

We multiply out each side:

$$\begin{pmatrix} -5a - 6b \\ 3a + 4b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Which gives us two simultaneous equations:

$$-5a - 6b = a$$

$$3a + 4b = b$$

We rearrange to get:

$$-6a = 6b$$

$$3a = -3b$$

Which both give us

$$a = -b$$

So, an eigenvector for $\lambda_1 = 1$ could be $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. There are many solutions to the equation $a = -b$, for example $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 300 \\ -300 \end{pmatrix}$. All of these are correct eigenvectors for $A$.

For $\lambda_2 = -2$:

$$\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -2 \begin{pmatrix} a \\ b \end{pmatrix}$$

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We multiply out each side:

$$\begin{pmatrix} -5a-6b \\ 3a+4b \end{pmatrix} = \begin{pmatrix} -2a \\ -2b \end{pmatrix}$$

Which gives us the simultaneous equations:

$$-5a-6b=-2a$$

$$3a+4b=-2b$$

Which we rearrange to get:

$$-6b=3a$$

$$3a=-6b$$

So, we have:

$$a=-2b$$

So, an eigenvector for $\lambda_2=-2$ could be $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

We can check we have found the right answer by calculating $\begin{pmatrix} -5a-6b \\ 3a+4b \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -5a-6b \\ 3a+4b \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and seeing if they are equal to $1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $-2\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ respectively.

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## Diagonalisation

Now that we have the eigenvectors and eigenvalues for a matrix, we can use a really cool property of matrices, to find the diagonalistion of the matrix.

Matrices can be diagonalised as follows:

$$A = PDP^{-1}$$

Where $A$ is the matrix we wish to diagonalise, $D$ is a diagonal matrix found from the eigenvalues of $A$ and $P$ is a matrix formed from the eigenvectors of $A$.

For example, when $\mathrm{A} = \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}$, we find that

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

A matrix with the eigenvalues on the diagonals and zeros elsewhere.

We find $P$ by writing the eigenvectors of $A$ as columns of $P$, in the same order as the eigenvalues they correspond to in $D$ (so in this case, an eigenvector corresponding to $\lambda_1 = 1$ would go in the first column, and an eigenvector corresponding to $\lambda_2 = -2$ goes in the second column):

$$P = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$$

We find $P^{-1}$ using your preferred method for finding an inverse:

$$P^{-1} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

So, we have found that

$$A = PDP^{-1}$$

$$\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

Now we have the cool bit:

If you were asked to calculate $A^6$, you could probably do this, but it would take a really long time, and it'd be very easy to make a mistake somewhere. However, if we look at $A^6$ as:

$$A^6 = (PDP^{-1})^6$$

we find that we can actually just calculate $A^6 = PD^6P^{-1}$.

This comes from the following:

$$A = PDP^{-1}$$
$$A^2 = PDP^{-1}PDP^{-1}$$
$$A^2 = PDIDP^{-1}$$
$$A^2 = PDDP^{-1}$$
$$A^2 = PD^2P^{-1}$$

This works again as we go to finding $A^3$:

$$A^3 = A^2A = PD^2P^{-1}PDP^{-1}$$
$$A^3 = PD^2IDP^{-1}$$
$$A^3 = PD^2DP^{-1}$$
$$A^3 = PD^3P^{-1}$$

and so on and so on, until we can see that $A^n = PD^nP^{-1}$.

So, for example, find $\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}^6$:

$$\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}^6 = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}^6\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}^6 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}^2\right)^3$$
$$\left(\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}^2\right)^3 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^3$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix}$$

So, we have

$$\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}^6 = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}^6 = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -64 & -64 \end{pmatrix}$$

$$\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}^6 = \begin{pmatrix} 127 & 126 \\ -63 & -62 \end{pmatrix}$$

Clearly, we aren't saying that this is a short calculation, but it is a much shorter (and more straightforward) calculation than multiplying out

$$A^6 = \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}$$

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