



What is a basis?

(Informally) a basis of \mathbb{R}^n is a set of vectors in \mathbb{R}^n that you can make any vector in \mathbb{R}^n out of by finding a linear combination of the basis vectors. For example,

 $\binom{1}{0}$ and $\binom{0}{1}$ are a basis for R^2 because we can write any vector in R^2 as a linear combination of the vectors, as $\binom{a}{b} = a \binom{1}{0} + b \binom{0}{1}$.

Why find a basis?

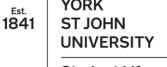
As with most things we do in linear algebra, finding a basis is a great way to condense a lot of information into a much smaller package. Imagine listing out every vector in R^3 , versus simply saying 'every vector in R^3 can be written as a linear combination of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Checking if a set of vectors is a basis

When we are given a set of vectors and asked 'is this set of vectors a basis in S?' we are being asked to check two things:

- Are the vectors linearly independent?
- Do they span S?

If the answer to both is yes, then we have a basis for *S*.



Student Life

Library and Learning Services



Linear independence

A set of vectors being linearly independent means that we cannot form one of the vectors as a linear combination of the others. A linear combination of a set of vectors $v_1, v_2, ..., v_n$ is $a_1v_1 + a_2v_2 + \cdots + a_nv_n$ where $a_1, a_2, ..., a_n$ are scalars.

For example,

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$
, $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ are not linearly independent, since $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$.

We can test for linear independence in a couple of different ways:

1. A set of vectors $v_1, v_2, ..., v_n$ are linearly independent if the only solution to $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$ is $a_1 = a_2 = ... = a_n = 0$.

For example, for the vectors $\binom{1}{2}$ and $\binom{3}{1}$, we find

$$a\begin{pmatrix}1\\2\end{pmatrix} + b\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

(for scalars a and b). Multiplying out, we get

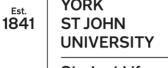
$$\binom{a}{2a} + \binom{3b}{b} = \binom{0}{0}$$

and so, we have

$$(1) a + 3b = 0$$

(2)
$$2a + b = 0$$

Solving these simultaneously gives us (2) - (1): -5b = 0 and so, b = 0. Giving (1): a + 0 = 0, and so a = 0. Therefore, the only solution we have is a = b = 0, so the vectors are linearly independent.





Library and Learning Services



2. The determinant of a matrix is 0 if and only if the columns are linearly dependent.

To use this method, we write the vectors as the columns of a matrix and then find the determinant.

For example, to test if $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ are linearly independent, we find:

$$det \begin{pmatrix} 1 & 3 & 0 \\ 0 & 3 & 0 \\ 2 & 4 & 2 \end{pmatrix} = 6 \neq 0$$

Therefore, the vectors are linearly independent.

Spanning

When we say a set of vectors $v_1, v_2, ..., v_n$ span S, we mean that we can find any vector v in S by writing it as $a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$ for scalars $a_1, a_2, ..., a_n$.

There is a very useful result that 'n linearly independent vectors will span \mathbb{R}^{n} ' (proof: see below in the method for showing independence and span in one step).

So, if you are testing whether your vectors span \mathbb{R}^n , and you have n linearly independent vectors, we know that they span \mathbb{R}^n .

Library and Learning Services

Study Development



Student Life

Library and Learning Services

Finding a basis for R^n

Study Development Factsheet

Showing we have a basis in one step

We may rewrite $a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$ as $(a_1 \quad a_2 \quad \cdots \quad a_n)V = v$, where V is a matrix whose columns are v_1, v_2, \ldots, v_n . If V is invertible, we have

$$(a_1 \quad a_2 \quad \dots \quad a_n)V = v$$
 $(a_1 \quad a_2 \quad \dots \quad a_n)VV^{-1} = vV^{-1}$
 $(a_1 \quad a_2 \quad \dots \quad a_n)I_n = vV^{-1}$
 $(a_1 \quad a_2 \quad \dots \quad a_n) = vV^{-1}$

Therefore, if V^{-1} exists, we know that there is always a solution for $a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$, and so S is spanned by v_1, v_2, \ldots, v_n .

Note: if V is invertible, this also means $v_1, v_2, ..., v_n$ are linearly independent, so we have also shown that $v_1, v_2, ..., v_n$ is a basis in this one calculation.

We also know that a matrix is invertible if and only if its determinant is not 0.

Therefore, to show that we have a basis:

Write the vectors as the columns of a matrix, find the determinant. If the determinant is 0, we do not have a basis. If the determinant is not 0, we have a basis.

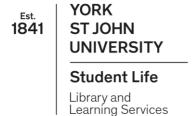
For example, to test if the vectors $\begin{pmatrix} 1\\4\\1 \end{pmatrix}$, $\begin{pmatrix} 0\\2\\1 \end{pmatrix}$ and $\begin{pmatrix} 1\\3\\0 \end{pmatrix}$ are a basis of \mathbb{R}^3 , we perform the following calculation:

$$det\begin{pmatrix} 1 & 0 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} = -1 \neq 0$$

Therefore, they are a basis of R^3 .

Library and Learning Services

Study Development





Finding a basis

Now that we know how to check if vectors are a basis, finding a basis is a pretty simple task. We know that we need to find the columns of an nxn matrix such that its determinant isn't zero.

For example, if we are asked to find a basis for R^3 we can simply fill in the blanks of a 3x3 matrix.

Start by choosing any vector in R^3 . We'll call this $v_1 = \begin{pmatrix} v_{1,1} \\ v_{1,2} \\ v_{1,3} \end{pmatrix}$ Then, choose a second vector in R^3 -

as long as this isn't a scalar multiple of your first vector it will work. We write this as v_2 =

$$\begin{pmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{pmatrix}$$
Next, we write

$$\begin{pmatrix} v_{1,1} & v_{2,1} & v_{3,1} \\ v_{1,2} & v_{2,2} & v_{3,2} \\ v_{1,3} & v_{2,3} & v_{3,3} \end{pmatrix}$$

We then find the determinant and choose values for $v_{3,1}$, $v_{3,2}$ and $v_{3,3}$ such that the determinant is not 0.

If we are finding a basis for \mathbb{R}^n with n > 3, we have to check for linear independence each time we add a new vector past the second vector added.

Hint: there are vectors that we sometimes call the 'standard basis vectors'. These are vectors that have only one non-zero entry that is 1. These are usually the easiest way to find a basis. For

example, for R^3 , the 'standard basis vectors' are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. If you are unsure, try

selecting the first two basis vectors in this format, and it should make it much easier to find a third vector.

Library and Learning Services

Study Development



Student Life

Library and Learning Services

Finding a basis for \mathbb{R}^n Study Development Factsheet

For example, find a basis for R^4 :

First, we choose $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ as one of the vectors. Since $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is not a scalar multiple of our first vector,

we choose this as our second vector. We select $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ as a potential third vector, but we must first

check it that these three vectors are linearly independent:

$$a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ gives } \begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ so we have } a = b = c = 0 \text{ as the only solution,}$$

and so they are linearly independent.

Finally, we set up:

$$\begin{pmatrix} 1 & 0 & 0 & v_{4,1} \\ 0 & 1 & 0 & v_{4,2} \\ 0 & 0 & 1 & v_{4,3} \\ 0 & 0 & 0 & v_{4,4} \end{pmatrix}$$

We find the determinant:

$$det \begin{pmatrix} 1 & 0 & 0 & v_{4,1} \\ 0 & 1 & 0 & v_{4,2} \\ 0 & 0 & 1 & v_{4,3} \\ 0 & 0 & 0 & v_{4,4} \end{pmatrix} = 1 \left(1 \left(v_{4,4} \right) \right) = v_{4,4}$$

Therefore, we have that $v_{4,1}$, $v_{4,2}$ and $v_{4,3}$ can be any values in R that we'd like, and $v_{4,4}$ can be

anything in R except 0. So, we could have a basis of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$.

Library and Learning Services

Study Development



YORK ST JOHN UNIVERSITY

Student Life

Library and Learning Services



Support: Study Development offers workshops, short courses, 1 to 1 and small group tutorials.

- Join a tutorial or workshop on the <u>Study Development tutorial and workshop webpage</u> or search 'YSJ study development tutorials.'
- Access our Study Success resources on the <u>Study Success webpage</u> or search 'YSJ study success.'

Library and Learning Services

Study Development

Email: studydevelopment@yorksj.ac.uk

Est. YORK 1841 ST JOHN