Matrices

Study Development Factsheet

Notation

A : any capital letter can be used to denote a matrix, but it must be explicitly stated that A is a matrix.

$M_{a \times b}$: a matrix with a rows and b columns.

$R^{m \times n}$: a matrix with all real entries, m rows and n columns.

Special matrices

The identity matrix (I_n): this is a matrix with 1s on the diagonal and 0s elsewhere, for example:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix has the special property that any matrix or vector multiplied by the identity matrix does not change, so $I_n A = A$ (similar to how multiplying a number by 1 doesn't change the value of the number).

The zero matrix (0): a matrix with all zero entries. Any matrix multiplied by a zero matrix becomes a zero matrix.

Operations

Addition:

Matrices can only be added if they are the same size. We add them by adding up numbers that are in the same position, for example:

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 9 & 4 \end{pmatrix} + \begin{pmatrix} 6 & 5 & 8 \\ 0 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 1+6 & 0+5 & 3+8 \\ 2+0 & 9+1 & 4+7 \end{pmatrix} = \begin{pmatrix} 7 & 5 & 11 \\ 2 & 10 & 11 \end{pmatrix}$$

Subtraction is performed similarly, for example:

$$\begin{pmatrix} 4 & 6 \\ 7 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 4-1 & 6-3 \\ 7-2 & 1-5 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 5 & -4 \end{pmatrix}$$

Scalar multiplication:

When we multiply a matrix by a scalar (a number or value that is not a matrix, for example 2 or a) we simply multiply each entry in the matrix by the scalar. For example:

$$p\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} pa & pb \\ pc & pd \end{pmatrix}$$

Multiplication:

Matrices can only be multiplied if the number of columns in the left matrix is equal to the number of rows in the right matrix.

The resulting matrix will have the same number of rows as the left matrix and the number of columns of the right matrix.

To multiply two matrices, we multiply the entries in the rows of the left matrix by the columns of the right matrix, as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & cf + bh \\ ce + dg & cf + dh \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} g & h \\ i & j \\ k & l \end{pmatrix} = \begin{pmatrix} ag + bi + ck & ah + bj + cl \\ dg + ei + fk & dh + ej + fl \end{pmatrix}$$

Note: when multiplying scalars, the order doesn't matter (e.g. $3 \times 2 = 2 \times 3$), however for matrices it matters greatly. In most cases,

$$AB \neq BA$$

Division:

We cannot divide by a matrix. There are ways to get around this issue sometimes (look at inverses) but dividing by a matrix is not possible.

Transpose:

Denoted A^T, the transpose of a matrix is formed by turning the rows of the matrix into columns and the columns into rows. For example:

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 5 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 3 & 2 \\ 2 & 5 \end{pmatrix}$$

Transpose properties:

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

$(cA)^T = cA^T$ (for a scalar c)

$(A^T)^{-1} = (A^{-1})^T$ (where A^{-1} is the inverse of A)

Trace:

The trace of a square matrix, $\text{tr}(A)$ is the sum of the entries on the diagonal. For example:

$$\text{tr}\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} = 3 + 2 = 5$$

Trace properties:

$\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$

$\text{tr}(AB) = \text{tr}(BA)$

$\text{tr}(cA) = c\,\text{tr}(A)$ (where c is a scalar)

$\text{tr}(A^T) = \text{tr}(A)$

$\text{tr}(A^T B) = \text{tr}(AB^T) = \text{tr}(B^T A) = \text{tr}(BA^T)$

Inverse of a matrix:

We can only find an inverse of a square matrix.

The inverse of a matrix A is denoted A^{-1} and holds the special property that $AA^{-1} = A^{-1}A = I_n$.

This is how we can sometimes "divide" by a matrix, by using its inverse. For example, if we want to solve the following equation for x, and we know an inverse of A exists:

$$Ax = B$$

We left multiply both sides by A^{-1}

$$A^{-1}Ax = A^{-1}B$$

Which gives us

$$I_n x = A^{-1}B$$

Which is the same as

$$x = A^{-1}B$$

There are many different ways to find an inverse of a matrix. For smaller matrices, we can use the $AA^{-1} + I_n$ property, and then solve for A^{-1}. For example:

To find the inverse of the matrix $\begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}$, we substitute in $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and solve for a, b, c, d:

$$AA^{-1} = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So, we have $\begin{pmatrix} a & b \\ 4a + 3c & 4b + 3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which we solve to get

$a = 1, \qquad b = 0$

$4(1) + 3c = 0$, and so $c = \frac{-4}{3}$

$4(0) + 3d = 1$, and so $d = \frac{1}{3}$.

Therefore, $A^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{4}{3} & \frac{1}{3} \end{pmatrix}$, or $A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ -4 & 1 \end{pmatrix}$.

There are lots of different ways to find an inverse of a matrix. For smaller matrices, this method is quick and simple, but as the matrices get larger it gets more time consuming to use this method, so you may want to find another method that you prefer.

Determinant:

We can only find a determinant for a square matrix.

The determinant of a matrix is a number that tells us certain properties of the matrix.

To calculate the determinant of a 2x2 matrix, we find:

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

For a 3x3, we find:

$$det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\left(det\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right) - d\left(det\begin{pmatrix} b & c \\ h & i \end{pmatrix}\right) + g\left(det\begin{pmatrix} b & c \\ e & f \end{pmatrix}\right)$$

For higher order determinants, we follow the same pattern:

1. Choose either a row or column in the matrix to work from.

   In the 3x3, we chose the first column, but we could have used any of them. A good idea is to pick the one with the most zeros in it.

2. Choose the first entry in the row or column. Ignore the entries that are in the same row and column as this entry and find the determinant of the leftover entries. For example:

   We choose the top row, so we focus on the first entry a and disregard the other numbers in its row and column (shown in red):

   $$\begin{pmatrix} \color{red}{a} & \color{red}{b} & \color{red}{c} \\ \color{red}{d} & e & f \\ \color{red}{g} & h & i \end{pmatrix}$$

   So, we find the determinant of the leftover entries:

   $$det\begin{pmatrix} e & f \\ h & i \end{pmatrix}$$

3. Multiply the entry we are focused on by the determinant we found:

   $$a\left(\det\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right)$$

4. Repeat this for each entry in your chosen row or column.

5. To calculate the final determinant, we add or subtract the entries multiplied by their sub-determinants in the following way:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & \cdots & \\ & & & \ddots \end{pmatrix}$$

where we alternate +'s and –'s.

This means that if the entry you chose to be multiplied by the sub-determinant is in a square with a + we add the entry multiplied by the sub-determinant, and if the entry was in a position with a – then we subtract the entry multiplied by the sub-determinant.

If this seems like a lot of calculation to do, that's because it is. Generally, if you are calculating a determinant of a matrix larger than a 3x3 it would be better to use an online determinant calculator. This saves a lot of time, and greatly reduces the risk of making a mistake in the calculation.

Determinant properties:

A matrix is invertible if and only if its determinant is not zero.

If the determinant of a matrix is zero, this means that the rows or columns are not linearly independent.

$det(I) = 1$

$det(cA) = c^n \, det(A)$ (for an $n \times n$ matrix)

$det(A^T) = det(A)$

$det(AB) = det(A) \, det(B)$

$$det(A^{-1}) = \frac{1}{det(A)}$$

If we swap two columns with each other in the matrix, we multiply the determinant by -1.

Adding a scalar multiple of one row to another row does not change the value of the determinant.

If there is a row or column of the matrix that is all zeros, then the determinant is 0.

Solving simultaneous equations using matrices

Simultaneous equations with more than two variables quite quickly become tricky and time consuming to solve. We can use a method of inverting a matrix to find a solution.

For example, we will use a matrix method to solve the following system:

$$3x + 2y + z = 8$$

$$-x + 4y + 3z = 2$$

$$x - 3y - 3z = -1$$

We want to set up an equation of the form $Ax = b$ where A is a 3x3 matrix, and v and b are 3x1 vectors.

We set up A by filling the matrix with the coefficients of the equations. The coefficients of x go into the first column, the coefficients of y go in the second and the coefficients of z go in the third column. Coefficients from the same equation need to go in the same row. So, for our example we find

$$A = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 4 & 3 \\ 1 & -3 & -3 \end{pmatrix}$$

We set up v by writing our three variables in order of the columns in A that they correspond to, so we have

$$v = \begin{pmatrix} x \\ \mathrm{y} \\ z \end{pmatrix}$$

Finally, we set up b by entering the numbers on the right-hand side of the equations in order corresponding to the rows of A. We have

$$b = \begin{pmatrix} 8 \\ 2 \\ -1 \end{pmatrix}$$

Now we have our matrix system:

$$\begin{pmatrix} 3 & 2 & 1 \\ -1 & 4 & 3 \\ 1 & -3 & -3 \end{pmatrix} \begin{pmatrix} x \\ \mathrm{y} \\ \mathrm{z} \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -1 \end{pmatrix}$$

We solve this system by finding A^{-1}, and multiplying both sides by it, so that we have:

$$Av = b$$
$$A^{-1}Av = A^{-1}b$$
$$I_n v = A^{-1}b$$
$$v = A^{-1}b$$

By using the identity matrix method, we find that

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 3 & -3 & -2 \\ 0 & 10 & 10 \\ 1 & -11 & -14 \end{pmatrix}$$

Therefore, we have

$$v = \frac{1}{10}\begin{pmatrix} 3 & -3 & -2 \\ 0 & 10 & 10 \\ 1 & -11 & -14 \end{pmatrix}\begin{pmatrix} 8 \\ 2 \\ -1 \end{pmatrix}$$

We multiply this out to get

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

So, the solutions are $x = 2$, $y = 1$ and $z = 0$.

Transformations

We can describe a transformation using a matrix. These are some of the most common 2x2 matrix transformations.

Stretch/compression:

A stretch/compression in the x direction by a factor of k is represented by a matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$.

A stretch/compression in the y direction by a factor of k is represented by a matrix $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$.

Rotation:

A rotation of a position vector θ radians or degrees clockwise about the origin is represented by a matrix $\begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}$

Reflection:

A reflection of a position vector in the line $y = tan(\theta)$ is given by a matrix $\begin{pmatrix} cos(2\theta) & sin(2\theta) \\ sin(2\theta) & -cos(2\theta) \end{pmatrix}$

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