YORK ST JOHN UNIVERSITY
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Matrices

Study Development Worksheet

Questions

1. Calculate the following:

a. $\begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 7 \\ 2 & 9 & 3 \end{pmatrix} + \begin{pmatrix} -2 & 3 & 1 \\ 1 & -3 & 0 \\ -9 & 2 & 7 \end{pmatrix}$

b. $\begin{pmatrix} 3 & 4 & -2 \\ 2 & 3 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 7 & -0.5 \\ 13 & 1 & -2 \end{pmatrix}$

c. $\begin{pmatrix} 6 & 1 \\ 3 & 4 \\ -2 & 0.4 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 10 & -2 \\ -3.4 & 2 \end{pmatrix}$

d. $3\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$

e. $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}$

f. $\begin{pmatrix} 3 & 1 & -4 \\ -10 & 5 & 8 \end{pmatrix}\begin{pmatrix} 2 & 7 \\ 18 & 2 \\ -3 & -1 \end{pmatrix}$

g. $\begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

2. For the matrix $M = \begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}$ find:
   a. M^2
   b. M^3
   c. M^{-1}
   d. $det(M)$
   e. M^T
   f. $\mathrm{tr}(M)$

3. For the matrix $A = \begin{pmatrix} 1 & -1 & 4 \\ -2 & 3 & 1 \\ 0 & 2 & 7 \end{pmatrix}$ find:
   a. A^2
   b. A^{-1}
   c. $det(A)$
   d. A^T
   e. $\mathrm{tr}(A)$

4. Given that for a 3×3 matrix C, $det(C) = -1$, find:
   a. $det(5C)$
   b. $det(C^T)$
   c. $det\left(C\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -3 \\ 8 & -2 & 1 \end{pmatrix}\right)$
   d. $det(C^{-1})$

5. By finding the inverse of a matrix, solve the following systems of simultaneous equations:

   a. $-x + 2y = -2$

      $4x - 3y = \frac{21}{2}$

   b. $3a + 2b - c = 0$

      $-a - b + 3c = 1$

      $2a - 2b + 7c = 10$

Answers

1. Calculate:

a. $\begin{pmatrix} 2+-2 & 3+3 & -1+1 \\ 0+1 & 1+-3 & 7+0 \\ 2+-9 & 9+2 & 3+7 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 0 \\ 1 & -2 & 7 \\ 2 & 11 & 10 \end{pmatrix}$

b. $\begin{pmatrix} 5 & 11 & -2.5 \\ 15 & 4 & -3 \end{pmatrix}$

c. $\begin{pmatrix} 6-2 & 1-1 \\ 3-10 & 4--2 \\ -2+3.4 & 0.4-2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -7 & 6 \\ 1.4 & -1.6 \end{pmatrix}$

d. $\begin{pmatrix} 3 & 9 \\ 3 & 0 \end{pmatrix}$

e. $\begin{pmatrix} 1\times 2+3\times 2 & 1\times -1+3\times 3 \\ 2\times 2+1\times 2 & 2\times -1+1\times 3 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 6 & 1 \end{pmatrix}$

f. $\begin{pmatrix} 36 & 27 \\ 46 & -68 \end{pmatrix}$

g. $\begin{pmatrix} (1\times 4)+(3\times 1)+(0\times 2) \\ (-1\times 4)+(2\times 1)+(-1\times 2) \\ (0\times 4)+(1\times 1)+(-1\times 2) \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ -1 \end{pmatrix}$

2. For the matrix $M = \begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}$ we have:

   a. $M^2 = \begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -9 \\ -15 & 46 \end{pmatrix}$

   b. $M^3 = \begin{pmatrix} 1 & -9 \\ -15 & 46 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} -25 & -64 \\ 108 & 337 \end{pmatrix}$

   c. Using the identity matrix method:

$$MM^{-1} = I_2$$

$$\begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2a - c & 2b - d \\ 3a + 7c & 3b + 7d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This gives us two systems off simultaneous equations, which we need to solve:

(1) $2a - c = 1$

(2) $3a + 7c \ = 0$

(2) + 7 (1) : $17a = 7$

Therefore, $a = \frac{7}{17}$, which gives us $c = \frac{-3}{17}$

(3) $2b - d = 0$

(4) $3b + 7d \ = 1$

(4) + 7 (3) : $17\ b \ = 1$

Therefore, $b = \frac{1}{17}$, which gives us $d = \frac{2}{17}$.

So, we have $M^{-1} = \frac{1}{17}\begin{pmatrix} 7 & 1 \\ -3 & 2 \end{pmatrix}$

d. $det(M) = (2 \times 7) - (3 \times -1) = 14 + 3 = 17$

e. $M^T = \begin{pmatrix} 2 & 3 \\ -1 & 7 \end{pmatrix}$

f. $\mathrm{tr}(M) = 2 + 7 = 9$

3. For the matrix $A = \begin{pmatrix} 1 & -1 & 4 \\ -2 & 3 & 1 \\ 0 & 2 & 7 \end{pmatrix}$, we have:

a. $A^2 = \begin{pmatrix} 1 & -1 & 4 \\ -2 & 3 & 1 \\ 0 & 2 & 7 \end{pmatrix}\begin{pmatrix} 1 & -1 & 4 \\ -2 & 3 & 1 \\ 0 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 31 \\ -8 & 13 & 2 \\ -4 & 20 & 51 \end{pmatrix}$

b. Using the augmented matrix method:

$$\begin{pmatrix} 1 & -1 & 4 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 2 & 7 & 0 & 0 & 1 \end{pmatrix} \to R_1 = r_1 - 4r_2$$

$$\begin{pmatrix} 9 & -13 & 0 & 1 & -4 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 2 & 7 & 0 & 0 & 1 \end{pmatrix} \to R_2 = r_2 - \frac{1}{7}r_3$$

$$\begin{pmatrix} 9 & -13 & 0 & 1 & -4 & 0 \\ -2 & \frac{19}{7} & 0 & 0 & 1 & \frac{-1}{7} \\ 0 & 2 & 7 & 0 & 0 & 1 \end{pmatrix} \to R_1 = r_1 + \frac{91}{19}r_2$$

$$\begin{pmatrix} \frac{-11}{19} & 0 & 0 & 1 & \frac{15}{19} & \frac{-13}{19} \\ -2 & \frac{19}{7} & 0 & 0 & 1 & \frac{-1}{7} \\ 0 & 2 & 7 & 0 & 0 & 1 \end{pmatrix} \to R_1 = -\frac{19}{11} \times r_1$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{-19}{11} & \frac{-15}{11} & \frac{13}{11} \\ -2 & \frac{19}{7} & 0 & 0 & 1 & \frac{-1}{7} \\ 0 & 2 & 7 & 0 & 0 & 1 \end{pmatrix} \to R_2 = r_2 + 2r_1$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{-19}{11} & \frac{-15}{11} & \frac{13}{11} \\ 0 & \frac{19}{7} & 0 & \frac{-38}{11} & \frac{-19}{11} & \frac{171}{77} \\ 0 & 2 & 7 & 0 & 0 & 1 \end{pmatrix} \to R_2 = \frac{7}{19}r_2$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{-19}{11} & \frac{-15}{11} & \frac{13}{11} \\ 0 & 1 & 0 & \frac{-14}{11} & \frac{-7}{11} & \frac{9}{11} \\ 0 & 2 & 7 & 0 & 0 & 1 \end{pmatrix} \rightarrow R_3 = \mathrm{r}_3 - 2\mathrm{r}_2$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{-19}{11} & \frac{-15}{11} & \frac{13}{11} \\ 0 & 1 & 0 & \frac{-14}{11} & \frac{-7}{11} & \frac{9}{11} \\ 0 & 0 & 7 & \frac{28}{11} & \frac{14}{11} & \frac{-7}{11} \end{pmatrix} \rightarrow R_3 = \frac{1}{7} r_3$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{-19}{11} & \frac{-15}{11} & \frac{13}{11} \\ 0 & 1 & 0 & \frac{-14}{11} & \frac{-7}{11} & \frac{9}{11} \\ 0 & 0 & 1 & \frac{4}{11} & \frac{2}{11} & \frac{-1}{11} \end{pmatrix} \rightarrow R_3 = \frac{1}{7} r_3$$

Therefore, $A^{-1} = \frac{1}{11}\begin{pmatrix} -19 & -15 & 13 \\ -14 & -7 & 9 \\ 4 & 2 & -1 \end{pmatrix}$

c. $det(A) = det\left(\begin{pmatrix} 1 & -1 & 4 \\ -2 & 3 & 1 \\ 0 & 2 & 7 \end{pmatrix}\right) = 1\,det\left(\begin{pmatrix} 3 & 1 \\ 2 & 7 \end{pmatrix}\right) + 2\,det\left(\begin{pmatrix} -1 & 4 \\ 2 & 7 \end{pmatrix}\right) +$
$0\,det\left(\begin{pmatrix} -1 & 4 \\ 3 & 1 \end{pmatrix}\right)$

$$= 1(21 - 2) + 2(-7 - 14) + 0$$
$$= 20 - 42 = -22$$

d. $A^T = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 3 & 2 \\ 4 & 1 & 7 \end{pmatrix}$

e. $\text{tr}(A) = 1 + 3 + 7 = 11$

4. Given that for a 3×3 matrix C, $det(C) = -1$, find:

a. $det(5C) = 5^3\, det(C) = 125 \times -1 = -125$

b. $det(C^T) = det(C) = -1$

c. $det\left(C\begin{pmatrix}1 & 3 & 1\\ 0 & 1 & -3\\ 8 & -2 & 1\end{pmatrix}\right) = det(C)\, det\begin{pmatrix}1 & 3 & 1\\ 0 & 1 & -3\\ 8 & -2 & 1\end{pmatrix} = (-1)(-85) = 85$

d. $det(C^{-1}) = \frac{1}{det(C)} = \frac{1}{-1} = -1$

5. By finding the inverse of a matrix, solve the following systems of simultaneous equations:

   a. $-x + 2y = -2$

   $4x - 3y = \frac{21}{2}$

   We can put these equations into the format $Ax = B$:

$$\begin{pmatrix} -1 & 2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{21}{2} \end{pmatrix}$$

   We then find A^{-1} (using any appropriate method, this example uses the augmented matrix method):

$$\begin{pmatrix} -1 & 2 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{pmatrix} \to \mathrm{R}_2 = \mathrm{r}_2 + 4\mathrm{r}_1$$

$$\begin{pmatrix} -1 & 2 & 1 & 0 \\ 0 & 5 & 4 & 1 \end{pmatrix} \to R_2 = \frac{1}{5}r_2$$

$$\begin{pmatrix} -1 & 2 & 1 & 0 \\ 0 & 1 & \frac{4}{5} & \frac{1}{5} \end{pmatrix} \to R_1 = r_1 - 2r_2$$

$$\begin{pmatrix} -1 & 0 & \frac{-3}{5} & \frac{-2}{5} \\ 0 & 1 & \frac{4}{5} & \frac{1}{5} \end{pmatrix} \to R_1 = -r_1$$

$$\begin{pmatrix} 1 & 0 & \frac{3}{5} & \frac{2}{5} \\ 0 & 1 & \frac{4}{5} & \frac{1}{5} \end{pmatrix}$$

   Therefore, $A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$.

   We can then rearrange $Ax = B$ to get $A^{-1}Ax = A^{-1}B$, which gives $Ix = A^{-1}B$, and so $x = A^{-1}B$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ \frac{21}{2} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{1}{2} \end{pmatrix}$$

So, the solution is $x = 3$, $y = \frac{1}{2}$.

b. $3a + 2b - c = 0$

$-a - b + 3c = 1$

$2a - 2b + 7c = 10$

$Ax = B$:

$$\begin{pmatrix} 3 & 2 & -1 \\ -1 & -1 & 3 \\ 2 & -2 & 7 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 10 \end{pmatrix}$$

We find the inverse of A, using the augmented matrix method:

$$\begin{pmatrix} 3 & 2 & -1 & 1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 1 & 0 \\ 2 & -2 & 7 & 0 & 0 & 1 \end{pmatrix} \to R_3 = r_3 + 2r_2$$

$$\begin{pmatrix} 3 & 2 & -1 & 1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 1 & 0 \\ 0 & -4 & 13 & 0 & 2 & 1 \end{pmatrix} \to R_1 = r_1 + 2r_2$$

$$\begin{pmatrix} 1 & 0 & 5 & 1 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 0 \\ 0 & -4 & 13 & 0 & 2 & 1 \end{pmatrix} \to R_2 = r_2 + r_1$$

$$\begin{pmatrix} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & -1 & 8 & 1 & 3 & 0 \\ 0 & -4 & 13 & 0 & 2 & 1 \end{pmatrix} \to R_3 = r_3 - 4r_2$$

$$\begin{pmatrix} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & -1 & 8 & 1 & 3 & 0 \\ 0 & 0 & -19 & -4 & -10 & 1 \end{pmatrix} \to R_3 = \frac{-1}{19}r_3$$

$$\begin{pmatrix} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & -1 & 8 & 1 & 3 & 0 \\ 0 & 0 & 1 & \frac{4}{19} & \frac{10}{19} & \frac{-1}{19} \end{pmatrix} \to R_2 = r_2 - 8r_3$$

$$\begin{pmatrix} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & -1 & 0 & \frac{-13}{19} & \frac{-23}{19} & \frac{8}{19} \\ 0 & 0 & 1 & \frac{4}{19} & \frac{10}{19} & \frac{-1}{19} \end{pmatrix} \to R_2 = -1r_2$$

$$\begin{pmatrix} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & 1 & 0 & \frac{13}{19} & \frac{23}{19} & \frac{-8}{19} \\ 0 & 0 & 1 & \frac{4}{19} & \frac{10}{19} & \frac{-1}{19} \end{pmatrix} \rightarrow R_1 = r_1 - 5r_3$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{-1}{19} & \frac{-12}{19} & \frac{5}{19} \\ 0 & 1 & 0 & \frac{13}{19} & \frac{23}{19} & \frac{-8}{19} \\ 0 & 0 & 1 & \frac{4}{19} & \frac{10}{19} & \frac{-1}{19} \end{pmatrix}$$

Therefore, $A^{-1} = \frac{1}{19}\begin{pmatrix} -1 & -12 & 5 \\ 13 & 23 & -8 \\ 4 & 10 & -1 \end{pmatrix}$.

We then find $x = A^{-1}A$:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{19}\begin{pmatrix} -1 & -12 & 5 \\ 13 & 23 & -8 \\ 4 & 10 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 10 \end{pmatrix}$$

And so:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$