Simultaneous Equations

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Solving simultaneous equations gives you the point at which two (or more) functions meet. They allow us to mathematically model real-life scenarios to do things like optimise budgets, and answer questions that would normally take a lengthy 'trial and error' process to solve. For example, if you were trying to figure out how to split a food budget for a party between two kinds of snacks, whilst making sure everyone got at least one of them, you could use simultaneous equations to find the answer quickly.

Solving simultaneous equations

There are a few ways we can use to solve simultaneous equations. All of them will provide the same answer, so you can choose which one you like the most, or which one works best for a particular question.

Direct substitution

If you have an equation that gives a value for one variable (or can be rearranged to give you that) you can directly sub it into the other equation. For example:

$$x = 4y + 2$$

$$\frac{x}{2} + 1 = y + 3$$

Since we have an expression for x, we can just sub it directly into the second equation:

$$\frac{4y + 2}{2} + 1 = y + 3$$

Which becomes:

$$2y + 1 + 1 = y + 3$$

Which rearranges to give:

$$y = 1$$

and so

$$x = 4(1) + 2 = 6$$

Addition and subtraction

Sometimes it is easier to add or subtract equations from each other. We decide how to do this by looking at the coefficients of the variables. Our aim is to have one variable 'disappear' (e.g. if we have $4x$ in one equation, we want to find a way of adding the other equation to it so that we have $4x - 4x$ in the resulting equation, so there are no x's left). For example:

$$3s - 2 = 8t$$

$$s + 1 = 6t$$

We minus $3 \times$ the second equation from the first, to get:

$$3s - 2 - 3(s + 1) = 8t - 3(6t)$$

Which becomes:

$$3s - 3s - 2 - 3 = 8t - 18t$$

Which simplifies to:

$$-5 = -10t$$

and so

$$t = \frac{1}{2}$$

Which gives us

$$3s - 2 = 8\left(\frac{1}{2}\right)$$

$$3s = 6$$

$$s = 2$$

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Matrix method

If you are comfortable working with matrices and finding inverses, this is a speedy way to solve simultaneous equations, particularly when there are more than 2 variables.

For the simultaneous equations

$$3p + 2 = 2q - 1$$

$$p + 7 = 4q$$

We rearrange to have the constants on one side and the variables on the other:

$$3p - 2q = -3$$

$$p - 4q = -7$$

We then set up a system of matrices in this format:

$$\begin{pmatrix} 3 & -2 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -3 \\ -7 \end{pmatrix}$$

(Try mutliplying these out to show yourself that this is equivalent to the system of equations)

We then find an inverse for the 2x2 matrix- there are many ways to do this, see the factsheet on matrices for more information.

The inverse of $\begin{pmatrix} 3 & -2 \\ 1 & -4 \end{pmatrix}$ is $\begin{pmatrix} \frac{4}{10} & \frac{-2}{10} \\ \frac{1}{10} & \frac{-3}{10} \end{pmatrix}$. We left-multiply both sides of the equation to get:

$$\begin{pmatrix} \frac{4}{10} & \frac{-2}{10} \\ \frac{1}{10} & \frac{-3}{10} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{4}{10} & \frac{-2}{10} \\ \frac{1}{10} & \frac{-3}{10} \end{pmatrix} \begin{pmatrix} -3 \\ -7 \end{pmatrix}$$

Which simplifies to give:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{9}{5} \end{pmatrix}$$

and so

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{9}{5} \end{pmatrix}$$

Which gives us

$$p = \frac{1}{5}$$

$$q = \frac{9}{5}$$

Note: If you are looking at this method and thinking 'I have no idea what this means'- don't worry! Any of these three methods will work, just pick your favourite.

Example with three variables

We will now look at solving a set of simultaneous equations with three variables using our different methods. Note that we generally need as many equations as we have variables in order to be able to solve them.

$$2x + y - z = 6$$
$$-x + 3y + z = 0$$
$$x - 2y + 4z = -4$$

Substitution method:

We rearrange the third equation to get $x = 2y - 4z - 4$

We then substitute this into the other two equations to get:

$$2(2y - 4z - 4) + y - z = 6$$
$$-(2y - 4z - 4) + 3y + z = 0$$

Which simplify to give:

$$5y - 9z = 14$$
$$y + 5z = -4$$

We rearrange the new second equation to get $y = -5z - 4$ and substitute it into the new first equation, to get $5(-5z - 4) - 9z = 14$, which simplifies to $-34z = 34$, and so $z = -1$. We then use the new second equation to find $y + 5(-1) = -4$, which gives $y = 1$. Then we use the original third equation to find $x - 2(1) + 4(-1) = -4$, and so $x = 2$.

Addition/subtraction method:

We add the second and third equations to get

$$(-x+3y+z)+(x-2y+4z)=(0)+(-4)$$

Which simplies to

$$y+5z=-4$$

We add 2 × the second equation to the first to get

$$(2x+y-z)+2(-x+3y+z)=6+2(0)$$

Which simplifies to give

$$7y+z=6$$

We now have two new simultaneous equations:

$$y+5z=-4$$

$$7y+z=6$$

We now take away 5 × the new second equation from the new first equation, to get $(y+5z)-5(7y+z)=(-4)-5(6)$, which tidies up to give $-34y=-34$, so $y=1$. We then find (from the new first equation) $1+5z=-4$, so $z=-1$. Finally, we use the original third equation to get $x-2(1)+4(-1)=-4$, so $x=2$.

Matrix method:

We set up our matrix system:

$$\begin{pmatrix} 2 & 1 & -1 \\ -1 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ -4 \end{pmatrix}$$

We find an inverse for the 3x3 matrix of:

$$\frac{1}{34}\begin{pmatrix} 14 & -2 & 4 \\ 5 & 9 & -1 \\ -1 & 5 & 7 \end{pmatrix}$$

We left-multiply both sides to get:

$$\frac{1}{34}\begin{pmatrix} 14 & -2 & 4 \\ 5 & 9 & -1 \\ -1 & 5 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ -1 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{34}\begin{pmatrix} 14 & -2 & 4 \\ 5 & 9 & -1 \\ -1 & 5 & 7 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \\ -4 \end{pmatrix}$$

Which mutliplies out to get:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

So, we have $x = 2$, $y = 1$ and $z = -1$.

Some examples in context (how to do wordy problems)

Questions that use simultaneous equations come up a lot in the real world.

For example:

Two people want to save up for a shared car. They decide to try to save up £500 a month between them. One person has a higher salary than the other, so they agree to contribute £50 more than the other person. How much should they each save per month?

We model this problem using simultaneous equations. We say the person with the higher income saves $£x$ per month and the person with the lower income saves $£y$ per month.

Then we have the simultaneous equations:

$$x + y = 500$$

$$x = y + 50$$

We solve by substitution, to get $(y + 50) + y = 500$, which gives $2y = 450$. So, the person with the lower income must save £225 per month. We then find that $x = 225 + 50$, so the person with the higher income should save £275 per month.

In order to solve a 'wordy' problem, try the following steps:

1. Decide what the variables should be.
   Generally, the variables will be whatever they ask you to find- so in the example above, it was the amount being put into savings by each person. It could also be something like 'how many of each type of sweets should they buy?' in which case, you'd have x amount of one sweet, y amount of another, then possibly z amount of a third type of sweet and so on.
2. Set up your two equations using the information given.
3. Solve using your preferred method.
4. Remember, when writing your answers, you might need to put units in.

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