YORK ST JOHN UNIVERSITY Est. 1841

Student Life

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Differentiation

Study Development Worksheet

Simple differentiation

Differentiate:

1. $f(x) = x^2$
2. $f(x) = 3x^2 + 2x$
3. $f(x) = x^3 + 5x + 12$
4. $f(x) = x^{-2} + 13x^3 + 24 + x$
5. $f(x) = x^{12} + x^{\frac{3}{2}}$
6. $f(x) = \frac{x}{2} + x^6 + 12x^3$
7. $f(x) = \frac{3x^2}{5} + 4x^7 + x^0$
8. $f(x) = 12x^2 + 5x + \frac{13x^5}{5} + 12$

Common differentiation rules

Differentiate:

1. $f(x) = cos\,x + 2\,sin\,x$
2. $f(x) = cos\,x + tan\,x$
3. $f(x) = log(x + 3)$
4. $f(x) = log(x^4 + x)$
5. $f(x) = sin\,x + log(x)$
6. $f(x) = log(x^3 + 3x^2)$

The chain rule

Differentiate:

1. $f(x) = sin(x^2)$
2. $f(x) = (x^2 + 3)^2$
3. $f(x) = (2x + x^2)^3$
4. $f(x) = tan(3x)$
5. $f(x) = cos(x + 1)$

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The product rule

Differentiate:

1. $f(x) = x\,tan(x)$
2. $f(x) = (x+1)(2x+3)$
3. $f(x) = x^2\,sin(2x)$
4. $f(x) = 2xe^x$
5. $f(x) = (x^2+4)(3x+1)^2$

The quotient rule

Differentiate:

1. $h(x) = \frac{x}{sin\,x}$
2. $h(x) = \frac{1}{cos\,x}$
3. $h(x) = \frac{x}{x+1}$
4. $h(x) = \frac{x^2+2x}{3x^3+x}$
5. $h(x) = \frac{sin\,x}{x^2}$
6. $h(x) = \frac{log(x)}{x}$
7. $h(x) = tan(x)$

Equation of a tangent

Find the tangent of the function $f(x)$ at the point P when:

1. $f(x) = 2x^2 + 3x$ and $P = (1,5)$
2. $f(x) = cos(x)$ and $P = (0,1)$
3. $f(x) = tan(2x)$ and $P = \left(\frac{\pi}{2}, 0\right)$
4. $f(x) = log(x)$ and $P = (e^3, 3)$

Turning points

Find (and define) the turning points of the functions:

1. $f(x) = 3x^3 - 4x + 1$
2. $f(x) = 2x^2 + 5x$
3. $f(x) = log\left(\frac{2x}{x^2+1}\right)$ where $x > 0$.
4. $f(x) = sin\,x$

Answers

Simple differentiation

1. $f'(x) = 2x$
2. $f'(x) = 6x + 2$
3. $f'(x) = 3x^2 + 5$
4. $f'(x) = -2x^{-3} + 39x^2 + 1$
5. $f'(x) = 12x^{11} + \frac{3}{2}x^{\frac{1}{2}}$
6. $f'(x) = \frac{1}{2} + 6x^5 + 36x^2$
7. $f(x) = \frac{3x^2}{5} + 4x^7 + 1$

   $$f'(x) = \frac{6x}{5} + 28x^6$$
8. $f'(x) = 24x + 5 + 13x^4$

Common differentiation rules

1. $f'(x) = -\sin x + 2\cos x$
2. $f'(x) = -\sin x + \sec^2 x$
3. $f'(x) = \frac{1}{x+3}$
4. $f'(x) = \frac{4x^3+1}{x^4+x}$
5. $f'(x) = \cos x + \frac{1}{x}$
6. $f'(x) = \frac{3x^2+6x}{x^3+3x^2}$

   $= \frac{x(3x+6)}{x(x^2+3x)}$

   $= \frac{3x+6}{x^2+3x}$

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The chain rule

1. $\mathrm{f}(u) = \sin(u)$ with $u = x^2$.

   $\frac{\mathrm{d}f(u)}{\mathrm{d}u} = cos(u)$, and $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$.

   Therefore, $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 2x\, cos(u) = 2x\, cos(x^2)$.

2. $f(u) = u^2$ with $u = x^2 + 3$.

   $\frac{\mathrm{d}f(u)}{\mathrm{d}u} = 2u$, and $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$.

   Therefore, $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 2u \times 2x = 4xu = 4x(x^2 + 3)$.

3. $f(u) = u^3$ with $u = 2x + x^2$.

   $\frac{\mathrm{d}f(u)}{\mathrm{d}u} = 3u^2$, and $\frac{\mathrm{d}u}{\mathrm{d}x} = 2 + 2x$.

   Therefore,

   $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = (2 + 2x)(3u^2) = 3(2 + 2x)(2x + x^2)^2 = 6x^2(1 + x)(2 + x)^2$.

4. $f(u) = tan(u)$ with $u = 3x$.

   $\frac{\mathrm{d}f(u)}{\mathrm{d}u} = sec^2(u)$, and $\frac{\mathrm{d}u}{\mathrm{d}x} = 3$.

   Therefore, $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 3\, sec^2(u) = 3\, sec^2(3x)$.

5. $f(u) = cos(u)$ with $u = x + 1$.

   $\frac{\mathrm{d}f(u)}{\mathrm{d}u} = -\, sin(u)$, and $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$.

   Therefore, $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = -\, sin(u) \times 1 = -\, sin(x + 1)$.

The product rule

1. $f(x) = x$ and $g(x) = tan(x)$.

   $f'(x) = 1$ and $g'(x) = sec^2(x)$.

   Therefore, $h'(x) = (1 \times tan(x)) + (x \times sec^2(x)) = tan(x) + x\,sec^2(x)$.

2. $f(x) = x + 1$ and $g(x) = 2x + 3$.

   $f'(x) = 1$ and $g'(x) = 2$.

   Therefore, $h'(x) = \big(1 \times (2x + 3)\big) + \big(2 \times (x + 1)\big) = 2x + 3 + 2x + 2 = 4x + 5$.

   Of course, you could also do this by expanding the brackets to get $f(x) = 2x^2 + 5x + 3$ and then differentiating that to get $f'(x) = 4x + 5$.

3. $f(x) = x^2$ and $g(x) = sin(2x)$.

   $f'(x) = 2x$ and $g'(x) = 2\,cos(2x)$ (you can find this using the chain rule).

   Therefore,

   $$h'(x) = (2x\ \times sin\ (2x)) + (x\text{^}2\ \times 2cos\ (2x))\ =\ 2x(sin\ (2x) + x\ cos\ (2x)).$$

4. $f(x) = 2x$ and $g(x) = e^x$.

   $f'(x) = 2$ and $g'(x) = e^x$.

   Therefore, $h'(x) = 2xe^x + 2e^x = 2e^x(x + 1)$.

5. $f(x) = x^2 + 4$ and $g(x) = (3x + 1)^2$.

   $f'(x) = 2x$ and $g'(x) = 6(3x + 1)$ (you can find this using the chain rule).

   Therefore, $h'(x) = 6(x^2 + 4)(3x + 1) + 2x(3x + 1)^2 = (3x + 1)\big(6(x^2 + 4) + 2x(3x + 1)\big) = 2(3x + 1)(6x^2 + x + 12)$.

The quotient rule

1. $f(x) = x$ and $g(x) = sin(x)$.

   $f'(x) = 1$ and $g'(x) = cos(x)$.

   Therefore, $h'(x) = \frac{sin(x) - x\,cos(x)}{sin^2(x)}$.

2. $f(x) = 1$ and $g(x) = cos(x)$.

   $f'(x) = 0$ and $g'(x) = -sin(x)$.

   Therefore, $h'(x) = \frac{sin(x)}{cos^2(x)}$.

3. $f(x) = x$ and $g(x) = x + 1$.

   $f'(x) = 1$ and $g'(x) = 1$.

   Therefore, $h'(x) = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$.

4. $f(x) = x^2 + 2x$ and $g(x) = 3x^3 + x$.

   $f'(x) = 2x + 2$ and $g'(x) = 9x^2 + 1$.

   Therefore, $h'(x) = \frac{(2x+2)(3x^3+x)-(x^2+2x)(9x^2+1)}{(3x^3+x)^2} = \frac{-3x^2-12x+1}{(3x^2+1)^2}$.

5. $f(x) = sin\,x$ and $g(x) = x^2$.

   $f'(x) = cos\,x$ and $g'(x) = 2x$.

   Therefore, $h'(x) = \frac{x^2 cos\,x - 2x\,sin\,x}{x^4} = \frac{x\,cos\,x - 2\,sin\,x}{x^3}$.

6. $f(x) = log(x)$ and $g(x) = x$.

   $f'(x) = \frac{1}{x}$ and $g'(x) = 1$.

   Therefore, we have that $h'(x) = \frac{1 - log(x)}{x^2}$.

7. $h(x) = \frac{\sin x}{\cos x}$.

   $f(x) = \sin x$ and $g(x) = \cos x$.

   $f'(x) = \cos x$ and $g'(x) = -\sin x$.

   Therefore, we have that $h'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$.

Equation of a tangent

1. $f'(x) = 4x + 3$.

   So, the gradient at P is equal to $f'(1) = 7$, and we have the equation for the tangent as $y = 7x + c$.

   We find c:

   $5 = 7 + c$, so $c = -2$, and the equation of the tangent is $y = 7x - 2$.

2. $f'(x) = -\sin(x)$.

   So, the gradient at P is equal to $f'(0) = 0$, and we have the equation for the tangent as $y = c$.

   We find c:

   $1 = c$, so, the equation of the tangent is $y = 1$.

3. $f'(x) = 2\sec^2(2x)$ (you can find this by using the chain rule).

   So, the gradient at P is $f'\left(\frac{\pi}{2}\right) = 2$, and we have the equation for the tangent as $y = 2x + c$.

   We find c:

   $0 = \pi + c$, so $c = -\pi$, and the equation of the tangent is $y = 2x - \pi$.

4. $f'(x) = \frac{1}{x}$.

   So, the gradient at P is equal to $f'(e^3) = \frac{1}{e^3} = e^{-3}$, and we have the equation for the tangent as $y = e^{-3}x + c$.

   We find c:

   $3 = e^{-3}e^3 + c = e^0 + c = 1 + c$, so $c = 2$, and the equation of the tangent is $y = e^{-3}x + 2$.

Turning points

Find (and define) the turning points of the functions:

1. $f'(x) = 9x^2 - 4$.

   $0 = 9x^2 - 4$ gives us $x^2 = \frac{4}{9}$, and so we have $x_1 = \frac{2}{3}$ and $x_2 = \frac{-2}{3}$.

   We find the corresponding y values:

   $\mathrm{f}(x_1) = 3\left(\frac{2}{3}\right)^3 - 4\left(\frac{2}{3}\right) + 1 = \frac{-7}{9} = \mathrm{y}_1$ and $f(x_2) = 3\left(\frac{-2}{3}\right)^3 - 4\left(\frac{-2}{3}\right) + 1 = \frac{25}{9} = y_2$.

   So, the two turning points of $f(x)$ are $\left(\frac{2}{3}, \frac{-7}{9}\right)$ and $\left(\frac{-2}{3}, \frac{25}{9}\right)$.

   To define the turning points, we differentiate $f'(x)$ to get $f''(x) = 18\mathrm{x}$.

   We now find $f''(x_1) = 18\left(\frac{2}{3}\right) = 12$. Since this is greater than zero, we know that $\left(\frac{2}{3}, \frac{-7}{9}\right)$ is a minima. We find $f''(x_2) = 18\left(\frac{-2}{3}\right) = -12$. Since this is less than zero, we know that $\left(\frac{-2}{3}, \frac{25}{9}\right)$ is a maxima.

2. $f'(x) = 4x + 5$.

   $0 = 4x + 5$ gives us $x = \frac{-5}{4}$. We find the corresponding y value: $f\left(\frac{-5}{4}\right) = \frac{-25}{8}$. So, the turning point of $f(x)$ is $\left(\frac{-5}{4}, \frac{-25}{8}\right)$.

   To define the turning point, we differentiate $f'(x)$ to get $f''(x) = 4$. Since this is greater than zero at every point, we know that the turning point is a minima.

3. Call $\frac{2x}{x^2+1} = h(x)$. We find $h'(x)$ using the quotient rule:
$p(x) = 2x$ and $q(x) = x^2 + 1$, therefore $p'(x) = 2$ and $q'(x) = 2x$. So, we have that $h'(x) = \frac{2(1-x^2)}{(x^2+1)^2}$.

Since $\frac{\mathrm{d}\, log(h(x))}{\mathrm{d}x} = \frac{h'(x)}{h(x)}$, we have that $f'(x) = \frac{2(1-x^2)(x^2+1)}{2x(x^2+1)^2} = \frac{1-x^2}{x(x^2+1)}$.

$0 = \frac{1-x^2}{x(x^2+1)}$ gives us that $x^2 = 1$, so $x_1 = 1$ and $x_2 = -1$. Since it is stated that $x > 0$, we discard x_2. We find the corresponding y value:

$f(x_1) = log(1) = 0$, so, we have that the turning point of the function exists at $(1,0)$.

To define the turning point, we differentiate $f'(x)$ to get $f''(x) = \frac{x^4-4x^2-1}{x^2(x^2+1)^2}$ (using the quotient rule).

We find $f''(x_1) = -2$, so we have that the turning point is a maxima.

4. $f'(x) = cos\, x$.

$0 = cos\, x$ gives us that $x = cos^{-1}(0) = \frac{\pi}{2}$. We also have that x could be equal to $-\frac{\pi}{2}, \frac{3\pi}{2}$ and so on. In fact, x is equal to $\frac{\pi}{2} + n\pi$ for any integer n.

We find the corresponding y values:

$f\left(\frac{\pi}{2}\right) = 1$, and in fact, $f\left(\frac{\pi}{2} + n\pi\right) = 1$ for all even n.

$f\left(-\frac{\pi}{2}\right) = -1$, and $f\left(\frac{\pi}{2} + n\pi\right) = -1$ for all odd n.

So, the turning points of $f(x)$ exist at $\left(\frac{\pi}{2} + 2m\pi, 1\right)$ and $\left(\frac{\pi}{2} + (2m+1)\pi, -1\right)$ for all integer values of m.

To define these turning points, we differentiate $f'(x)$ to give $f''(x) = -sin(x)$.

$f''\left(\frac{\pi}{2} + 2m\pi\right) = -1$ and $f''\left(\frac{\pi}{2} + (2m+1)\pi\right) = 1$ for all integer values of m.

Therefore, we know that the turning points of the form $\left(\frac{\pi}{2} + 2m\pi, 1\right)$ are maxima and all turning points of the form $\left(\frac{\pi}{2} + (2m+1)\pi, -1\right)$ are minima.

Note: If you found the last two questions (or indeed any of the questions in this worksheet) to be difficult, don't worry! Differentiation takes a lot of practice, and this worksheet contains some quite tricky differentiation questions. If you are less confident with the later questions, try to focus on the simple differentiation section until you feel happy with that.

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