Differentiation

Study Development Factsheet

Differentiation is the process of finding the rate of change of a function. For example, if we differentiate a function that describes the speed of a car over an hour, we get an expression for the acceleration of the car.

Notation

There are a few different ways to denote that a function has been differentiated. When studying differentiation for a course, there will most likely be one form of notation that the course leaders will prefer, but otherwise the choice is left up to the individual to make. Some notations are more appropriate in certain circumstances than others.

Leibniz's notation

This notation is very commonly used, and is the most appropriate when using the chain rule.

First derivative: $\frac{\mathrm{d}y}{\mathrm{d}x}$ (may also be written as $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$, $\frac{\mathrm{d}f}{\mathrm{d}x}(\mathrm{x})$, $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$.)

$\frac{\mathrm{d}y}{\mathrm{d}x}$ means the derivative of the function y with respect to x.

Second derivative: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

nth derivative: $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$

nth derivative when $x = a$: $\frac{\mathrm{d}^\mathrm{n}y}{\mathrm{d}x^n}|_{x=a}$ (may also be written as $\frac{\mathrm{d}y}{\mathrm{d}x}\,(a)$.)

Lagrange's notation

This notation is also very commonly used. It is slightly less clear than Leibniz's notation, as it does not state which variable we are differentiating with respect to, but is more commonly used for things like Taylor and Maclaurin's theorem, or in situations where it is obvious that we are differentiating with respect to a certain variable.

First derivative of $f(x)$: $f'(x)$

Second derivative of $f(x)$: $f''(x)$

nth derivative of $f(x)$: $f^{\text{Roman numeral of n}}(x)$

First derivative of $f(x)$, evaluated at $x = a$: $f'(a)$

Euler notation

This notation is less common. It is used by some mathematicians as it is regarded as a clear notation, but as it is not used in many textbooks and papers, its popularity has remained low.

First derivative of $f(x)$: $\mathrm{D}f(x)$

Second derivative of $f(x)$: $\mathrm{D}^2 f(x)$

nth derivative of $f(x)$: $\mathrm{D}^n f(x)$

nth derivative of $f(x)$ when $x = a$: $\mathrm{D}^n f(a)$

Newton's notation

Newton's notation is occasionally used by physicists and engineers, but is not commonly favoured by most mathematicians as it is easily confused with the notation for a recurring decimal or a mean value.

First derivative of y: $\dot{y}$

Second derivative of y: $\ddot{y}$

The nth derivative is denoted by placing n dots above y.

Simple differentiation

If the function you wish to differentiate is of the form $f(x) = ax^n$ where a and n are constants, then we differentiate it as follows:

$$f'(x) = (a \times n)x^{n-1}$$

For example, when $f(x) = 3x^2$, $f'(x) = (3 \times 2)x^1 = 6x$.

For a function of the form $f(x) = ax^n + bx^m + \cdots$ we simply differentiate each term from left to right to get:

$$f'(x) = (a \times n)x^{n-1} + (b \times m)x^{m-1} + \cdots$$

For example, when $f(x) = 4x^3 + 12x^2 + 3x^{-1}$,

$f'(x) = (4 \times 3)x^2 + (12 \times 2)x^1 + (3 \times -1)x^{-2} = 12x^2 + 24x - 3x^{-2}$.

This will get easier with practice! There is a worksheet with practice questions available on the Study Success page.

Some differentials to look out for:

- The differential of a constant (e.g 2, 7, 0.003, etc) is 0.
- When differentiating a term of the form $f(x) = ax$ with respect to x, we follow the usual process: $f'(x) = (a \times 1)x^{1-1} = ax^0$. Using power rules, we know that $x^0 = 1$, therefore $f'(x) = a$.
- When differentiating a term of the form $f(x) = ax^{\frac{m}{n}}$ with respect to x, we follow the usual process: $f'(x) = \left(a \times \frac{m}{n}\right)x^{\left(\frac{m}{n}-1\right)} = \left(\frac{am}{n}\right)x^{\frac{m-n}{n}}$.

Common differentiation rules

$$\frac{\mathrm{d}\sin(x)}{\mathrm{d}x} = cos(x)$$

$$\frac{\mathrm{d}\cos(x)}{\mathrm{d}x} = -sin(x)$$

$$\frac{\mathrm{d}(-sin(x))}{\mathrm{d}x} = -cos(x)$$

$$\frac{\mathrm{d}(-cos(x))}{\mathrm{d}x} = sin(x)$$

$$\frac{\mathrm{d}\,tan(x)}{\mathrm{d}x} = sec^2(x)$$

Differential of natural log: $\frac{\mathrm{d}\,log\left(f(x)\right)}{\mathrm{d}x} = \frac{f'(x)}{f(x)}$

$$\frac{\mathrm{d}e^{f(x)}}{\mathrm{d}x} = f'(x)e^{f(x)}$$

The chain rule

The chain rule is as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y(u)}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

For example, for a function such as $f(x) = (3x+2)^2$, we write $f(u) = u^2$ with $u = 3x + 2$. We find:

$$\frac{\mathrm{d}f(u)}{\mathrm{d}u} = \frac{\mathrm{d}(u^2)}{\mathrm{d}u} = 2u$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}(3x+2)}{\mathrm{d}x} = 3$$

Now, we use the chain rule to find $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}f(u)}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2u \times 3 = 6u$$

Finally, we substitute $u = 3x + 2$:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 6u = 6(3x+2).$$

As a second example, consider $\mathrm{f(x)} = \sin(2\mathrm{x})$.

We write $f(u) = sin(u)$ with $u = 2x$.

We then find:

$$\frac{\mathrm{d}f(u)}{\mathrm{d}u} = \frac{\mathrm{d}\,sin(u)}{\mathrm{d}u} = cos(u)$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}(2x)}{\mathrm{d}x} = 2$$

Now, use the chain rule to find $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}f(u)}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = cos(u) \times 2 = 2\,cos(u).$$

Finally, we substitute $u = 2x$:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 2\,cos(u) = 2\,cos(2x).$$

The product rule

$$\frac{\mathrm{d}f(x)g(x)}{\mathrm{d}x} = f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x} + g(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x}$$

For example, for a function such as $h(x) = x\,sin(x)$, we find

$f(x) = x$, $g(x) = sin(x)$,

$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 1$, $\frac{\mathrm{d}g(x)}{\mathrm{d}x} = cos(x)$.

We then use the product rule:

$$\frac{\mathrm{d}h(x)}{\mathrm{d}x} = \frac{\mathrm{d}f(x)g(x)}{\mathrm{d}x} = f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x} + g(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x} = x \times cos(x) + sin(x) \times 1 = x\,cos(x) + sin(x)$$

The quotient rule

Until now, the rules have been written in Leibniz notation. Here, we swap to Lagrange notation to make the formula look less scary.

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

For example, for a function such as $h(x) = \frac{2x}{x+3}$, we find:

$f(x) = 2x,\qquad g(x) = x + 3,$

$f'(x) = 2,\qquad g'(x) = 1$

Now, we use the quotient rule to solve for $h'(x)$:

$$h'(x) = \left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2} =$$

$$\frac{((x+3) \times 2) - (2x \times 1)}{(x+3)^2} = \frac{2(x+3) - 2x}{(x+3)^2} = \frac{2x + 6 - 2x}{(x+3)^2} = \frac{6}{(x+3)^2}.$$

Equation of a tangent

One use of differentiation is to find the gradient of a function. A ‘tangent’ is a straight line that touches a curve at one point (though it may intersect again at some other points on the curve) and has the same gradient as the curve at the point it touches.

To find the equation of a tangent at a point on a function:

1. Differentiate the function. This gives you an expression for the gradient of the function.
2. Calculate the gradient using the expression from step 1 at the point you want the tangent at. Do this by plugging in the x value given.
3. Write a new expression of the form: $y = (\text{gradient at the point})x + c$.
4. Use the x and y values of the point given to calculate c.

For example, find the tangent to the function $f(x) = x^3 + 2x^2 + 1$ at the point $(1,4)$:

1. $f'(x) = 3x^2 + 4x$
2. $f'(1) = 3(1)^2 + 4(1) = 3 + 4 = 7$.
3. $y = 7x + c$.
4. $4 = 7(1) + c$

   Which implies that $-3 = c$

   Therefore, the equation of the tangent at (1,4) is $y = 7x - 3$.

Increasing or decreasing functions

What does the gradient actually mean?

- A positive gradient means the function is increasing. This means that y (or $f(x)$) increases as x increases.
- A negative gradient means the function is decreasing. So as x increases, y decreases.
- A larger positive gradient means that the function is increasing faster (ie y is increasing a lot each time x increases).
- A smaller positive gradient means that the function is increasing slower (ie y only increases a little bit each time x increases).
- A larger negative gradient means that the function is decreasing faster (ie y decreases a lot each time x increases).
- A smaller negative gradient means that the function is decreasing slower (ie y only decreases a little bit each time x increases).
- A gradient that is equal to 0 means that function is neither increasing or decreasing. y remains the same as x increases.

Turning points

A ‘turning point’ or ‘stationary point’ on a function is a point where the gradient of the function is equal to 0. These can be one of three types: a maxima, a minima, or a point of inflection. To find a turning point:

1. Differentiate the function.
2. Equate the differential to 0 and solve for x.
3. Substitute these x values into your original function to find the corresponding y values.

For example, find the turning points of the function $\mathrm{f}(x) = x^3 - \frac{3x^2}{2} - 36\mathrm{x} + 2$.

1. $f'(x) = 3x^2 - 3x - 36$
2. $3x^2 - 3x - 36 = 0$

   We factorise this to get: $(3x + 9)(x - 4) = 0$

   Then we solve for x: $x_1 = -3, x_2 = 4$.
3. $f(x_1) = (-3)^3 - \frac{3(3)^2}{2} - 36(3) + 2 = -146.5 = y_1$

   $$f(x_2) = (4)^3 - \frac{3(4)^2}{2} - 36(4) + 2 = -102 = y_2$$

   Therefore, the two turning points are at (-3,-146.5) and (4,-102).

Defining turning points

There are two ways to determine what kind of turning point you have. A 'maxima' is a point at which the function stops increasing and starts to decrease (as x increases). A 'minima' is a point at which a function stops decreasing and starts to increase (as x increases). A 'point of inflection' is one where the function has gradient 0, but is either increasing before and after or decreasing before and after.

Method 1: checking either side

1. Find a turning point.
2. Check the value of the gradient before the turning point and after (note: choose values to check that are close to the turning point. If you choose one that is so far away that there is another turning point between that point and the turning point you are checking then the test will not work).
3. If the value before is negative, and the value after is positive, you have a minima. If the value before is positive, and the value after is negative, you have a maxima. If the signs before and after are the same, it is a point of inflection.

Method 2: second differentiation

1. Find the turning points.
2. Differentiate the gradient function (ie find $f''(x)$).
3. For turning points $x_1, x_2, x_3, \ldots$, find $f''(x_1), f''(x_2), f''(x_3), \ldots$
4. If you get a positive value, you have a minima. A negative value is a maxima, and 0 is a point of inflection.

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