Complex numbers

Study Development Worksheet

Example

1. Convert $z = 3 + 4i$ into Polar form.
2. Convert $z = 5e^{i\frac{\pi}{2}}$ into Cartesian form.

Answer

1. We want to write $z = x + iy$ in the form $z = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$.

   We first find r:

   $r = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

   Note: -5 is also an answer to $\sqrt{25}$, however we usually write r to be positive as it is representing a length.

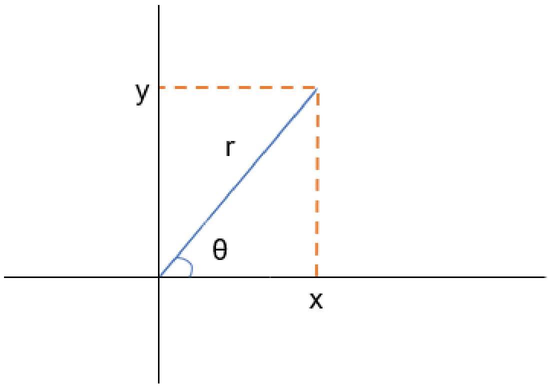
   Then, we find θ:

   $\theta = tan^{-1}\left(\frac{4}{3}\right) = 0.92729^c$

   Note: We always work in Radians when writing complex numbers in Polar form.

   Now, we plug in these values to get: $z = re^{i\theta} = 5e^{0.92729i}$.

2.



The Cartesian form ($z = x + iy$) is found using the formulae:

$x = r\,cos(\theta)$

$y = r\,sin(\theta)$

So, we have $x = 5\,cos\left(\frac{\pi}{2}\right) = 0$, and $y = 5\,sin\left(\frac{\pi}{2}\right) = 5$, which means that we have $z = 5i$.

Questions

1. Convert the following to Polar form:
   a. $z = 3 + i$
   b. $z = 4 + 4i$
   c. $z = 6 + 2\sqrt{3}i$
2. Convert the following to Cartesian form:
   a. $z = 4e^{\frac{i\pi}{3}}$
   b. $z = 2e^{\frac{i\pi}{4}}$
   c. $z = \sqrt{3}e^{\frac{i4\pi}{3}}$
3. Find the modulus and the argument of the following:
   a. $z = 5e^{2i}$
   b. $z = \sqrt{3} + i$
   c. $z = 9 + 9i$
   d. $z = 3 + 5i.$
4. Calculate the following:
   a. $(2 + 2i)(7 + 4i)$
   b. $(6 + 6i)^{10}$
   c. $\left(4\sqrt{3} + 4i\right)^5$
   d. $\left(10\sqrt{3} + 30i\right)^7$

Answers

1. We convert the values from the form $z = x + iy$ into the form $z = re^{i\theta}$ using $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$.

   a. $r = \sqrt{3^2 + 1^2} = \sqrt{10}$

   $\theta = tan^{-1}\left(\frac{1}{3}\right) = 0.32175^c$

   Therefore, $z = \sqrt{10}e^{0.32175i}$.

   b. $r = \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$

   $\theta = tan^{-1}\left(\frac{4}{4}\right) = \frac{\pi}{4}$

   Therefore, $z = 4\sqrt{2}e^{\frac{i\pi}{4}}$.

   c. $r = \sqrt{\left(2\sqrt{3}\right)^2 + 6^2} = \sqrt{12 + 36} = \sqrt{48} = 4\sqrt{3}$

   $$\theta = tan^{-1}\left(\frac{2\sqrt{3}}{6}\right) = \frac{\pi}{6}$$

   Therefore, $z = 4\sqrt{3}e^{\frac{i\pi}{6}}$.

2. We convert the values from the form $z = re^{i\theta}$ into the form $z = x + iy$ using $x = r\,cos(\theta)$ and $y = r\,sin(\theta)$.

   a. $y = 4\,sin\left(\frac{\pi}{3}\right) = 2\sqrt{3}$ and $x = 4\,cos\left(\frac{\pi}{3}\right) = 2$, so we have that $z = 2 + 2\sqrt{3}$.

   b. $y = 2\,sin\left(\frac{\pi}{4}\right) = \sqrt{2}$ and $x = 2\,cos\left(\frac{\pi}{4}\right) = \sqrt{2}$, so we have that $z = \sqrt{2} + \sqrt{2}i$.

   c. $\mathrm{y} = \sqrt{3}\,sin\left(\frac{4\pi}{3}\right) = \frac{-\sqrt{3}}{2}$ and $\mathrm{x} = \sqrt{3}\,cos\left(\frac{4\pi}{3}\right) = \frac{-1}{2}$, so we have that $z = \frac{-1}{2} - \frac{\sqrt{3}i}{2}$.

3. We find the modulus and argument of z using $\text{mod}(z) = \sqrt{x^2 + y^2} = \text{r}$ and $arg(z) = tan^{-1}\left(\frac{y}{x}\right) = \theta$.
   a. $\text{mod}(z) = r = 5$, $arg(z) = \theta = 2^c$.

   b. $\text{mod}(z) = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{4} = 2$, $arg(z) = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi^c}{6}$.

   c. $\text{mod}(z) = \sqrt{9^2 + 9^2} = 9\sqrt{2}$, $arg(z) = tan^{-1}\left(\frac{9}{9}\right) = \frac{\pi^c}{4}$.

   d. $\text{mod}(z) = \sqrt{3^2 + 5^2} = \sqrt{36} = 6$, $arg(z) = tan^{-1}\left(\frac{5}{3}\right) = 1.03038^c$.

4. When it would be time consuming to multiply many complex numbers together, we can convert them into Polar form and then perform the power operator.
   a. Using the FOIL method: $(2 + 2i)(7 + 4i) = 14 + 8i + 14i + 8i^2$. Since $i = \sqrt{-1}$, we know that $i^2 = -1$, so we have $14 + 8i + 14i + 8i^2 = 14 + 22i - 8 = 6 + 22i$.

   b. We first convert $6 + 6i$ into Polar form:

      $r = \sqrt{6^2 + 6^2} = 6\sqrt{2}$, and $\theta = tan^{-1}\left(\frac{6}{6}\right) = \frac{\pi}{4}$, so $6 + 6i = 6\sqrt{2}e^{\frac{i\pi}{4}}$.

      We then find $(6 + 6\text{i})^{10} = \left(6\sqrt{2}\text{e}^{\frac{\text{i}\pi}{4}}\right)^{10} = \left(6\sqrt{2}\right)^{10}\text{e}^{\frac{\text{i}10\pi}{4}} = 1934917632\text{e}^{\frac{\text{i}5\pi}{2}}$.

   c. We convert $4\sqrt{3} + 4i$ into Polar form:

      $r = \sqrt{\left(4\sqrt{3}\right)^2 + 4^2} = \sqrt{64} = 8$, and $\theta = tan^{-1}\left(\frac{4}{4\sqrt{3}}\right) = \frac{\pi}{6}$, so $4\sqrt{3} + 4i = 8e^{\frac{i\pi}{6}}$.

      We then find $\left(8e^{\frac{i\pi}{6}}\right)^5 = 8^5 e^{\frac{i5\pi}{6}} = 32768e^{\frac{i5\pi}{6}}$.

d. We convert $10\sqrt{3} + 30i$ into Polar form:

$r = \sqrt{\left(10\sqrt{3}\right)^2 + 30^2} = 20\sqrt{3}$, and $\theta = tan^{-1}\left(\frac{30}{10\sqrt{3}}\right) = \frac{\pi}{3}$, so $10\sqrt{3} + 30i = 20\sqrt{3}e^{\frac{i\pi}{3}}$.

We then find $\left(20\sqrt{3}e^{\frac{i\pi}{3}}\right)^7 = \left(20\sqrt{3}\right)^7 e^{\frac{i7\pi}{3}} = 59859675909.6e^{\frac{i7\pi}{3}}$.

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