Eigenvectors and Eigenvalues

Study Development Worksheet

Questions

1. Find the characteristic equations of the following matrices:

   a. $P = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

   b. $Q = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}$

   c. $R = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & -1 \end{pmatrix}$

   d. $S = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 9 \\ 1 & 0 & -1 \end{pmatrix}$

   e. $T = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

   For this question (and all matrices above dimension 3) most people would use an online solver. This question is thrown in for people who want a challenge, but you are welcome to use it just to practice using an online eigenvector and eigenvalue finder.

2. Using your answers to question 1, find the eigenvalues of the matrices:

   a. P

   b. Q

   c. R

   d. S

   e. T

3. Find an eigenvector corresponding to each of the eigenvalues of the matrices:

a. P

b. Q

c. R

d. S

e. T

Answers

1. Find the characteristic equations of the following matrices:

To find the characteristic equation of an n-dimensional matrix A, we find $det(A - \lambda I_n) = 0$:

a. $det\left(\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = det\left(\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) =$

$$det\begin{pmatrix} 1-\lambda & 3-0 \\ 2-0 & 2-\lambda \end{pmatrix} = det\begin{pmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - (3)(2)$$

$$= 2 - \lambda - 2\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 3\lambda - 4 = 0$$

b. $\lambda^2 + 4\lambda + 4 = 0$

c. $det\left(\begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & -1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) =$

$$det\left(\begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right) =$$

$$det\begin{pmatrix} -2-\lambda & 0 & 0 \\ 0 & 4-\lambda & -1 \\ 0 & 0 & -1-\lambda \end{pmatrix} =$$

$$(-2-\lambda)(4-\lambda)(-1-\lambda) = 0$$

You can leave it like this (since it's already factorised, which will help with question 2c, or if you expand and simplify this you'll get $-\lambda^3 + \lambda^2 + 10\lambda + 8 = 0$.

d. $-\lambda^3 - \lambda^2 + 4\lambda + 4 = 0$

e. $$det\left(\begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) =$$

$$det\left(\begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}\right) =$$

$$det\begin{pmatrix} 1-\lambda & 0 & 0 & 2 \\ 1 & 3-\lambda & 0 & 0 \\ 0 & 2 & -1-\lambda & 2 \\ 0 & 0 & 0 & 3-\lambda \end{pmatrix} =$$

$$(1-\lambda)(3-\lambda)(-1-\lambda)(3-\lambda) =$$

$$(1-\lambda)(3-\lambda)^2(-1-\lambda) = 0$$

You can leave it in this format since it's already factorised, which will help with 2e, or you can expand and simplify to get:

$$\lambda^4 - 6\lambda^3 + 8\lambda^2 + 6\lambda - 9 = 0$$

2. Using your answers to question 1, find the eigenvalues of the matrices:

   a. To find the eigenvalues, we solve the characteristic equation of the matrix. $-4 - 3\lambda + \lambda^2 = 0$ factorises to give $(\lambda - 4)(\lambda + 1) = 0$, and so the solutions/eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$.

   b. $\lambda^2 + 4\lambda + 4 = (\lambda + 2)(\lambda + 2) = 0$

      $\lambda = -2$

   c. $(\lambda + 1)(-\lambda + 4)(\lambda + 2) = 0$

      $\lambda_1 = -1,\ \lambda_2 = 4,\ \lambda_3 = -2$

   d. $-\lambda^3 - \lambda^2 + 4\lambda + 4 = (\lambda + 1)(-\lambda + 2)(\lambda + 2)$

      $\lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = -2$

e. $(\lambda - 1)(\lambda + 1)(\lambda - 3)^2$

$\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = 3$

3. Find an eigenvector of the matrices:

   a. To find an eigenvector corresponding each of the eigenvalues, we solve $\mathrm{P}\mathrm{x} = \lambda\mathrm{x}$ for each λ.

$$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

Which gives

$$\begin{pmatrix} x + 3y \\ 2x + 2y \end{pmatrix} = \begin{pmatrix} \lambda\, x \\ \lambda\, y \end{pmatrix}$$

If we write and these as simultaneous equations, we get

$$x\ +\ 3y\ =\ \lambda x$$

$$2x + 2y = \lambda y$$

For $\lambda_1 = 4$:

$$x\ +\ 3y\ =\ 4x$$

$$2x\ +\ 2y\ =\ 4y$$

Which both simplify to give

$$x = y$$

So, an eigenvector corresponding to $\lambda_1 = 4$ could be any scalar multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -1$, we have

$$x + 3y = -x$$

$$2x + 2y = -y$$

Which we rearrange to get

$$3y = -2x$$

So, an eigenvector corresponding to $\lambda_2 = -1$ could be any scalar multiple of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

   b. $Qx = \lambda x$

$$\begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -3x - y \\ x - y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

$-y = x$

Therefore, an eigenvector of Q is any scalar multiple of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

c. We find each solution to $Rx = \lambda x$:

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We multiply this out to get:

$$\begin{pmatrix} -2x \\ 4y - z \\ -z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$$

For $\lambda_1 = -1$:

$-2x = -x$, which implies that $x = 0$

$4y - z = -y$, which implies that $5y = z$

$-z = -z$, so we choose $z = 1$, which gives $y = \frac{z}{5} = \frac{1}{5}$

Therefore, an eigenvector of R which corresponds to $\lambda_1 = -1$ is any scalar multiple of $\begin{pmatrix} 0 \\ \frac{1}{5} \\ 1 \end{pmatrix}$.

For $\lambda_2 = 4$:

$-2x = 4x$, which implies that $x = 0$

$4y - z = 4y$, which implies that $z = 0$

$-z = 4z$, which has already been solved by $z = 0$

This means that y can take any real value, so an eigenvector of R corresponding to $\lambda_2 = 4$ is any scalar multiple of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Finally, for $\lambda_3 = -2$:

$-2x = -2x$, so $x = x$

$4y - z = -2y$, which implies that $6y = z$

$-z = -2z$, which implies that $z = 0$, which further implies that $y = 0$ since $6y = z$

So, an eigenvector of R corresponding to $\lambda_3 = -2$ is any scalar multiple of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

d. $Sx = \lambda x$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 9 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x + 3z \\ -y + 9z \\ x - z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$$

For $\lambda_1 = -1$:

$x + 3z = -x$, which rearranges to get $2x = -3z$

$-y + 9z = -y$, which implies that $z = 0$, which further implies that $x = 0$

$x - z = -z$, which is solved by the above $x = z = 0$.

So, an eigenvector of S corresponding to $\lambda_1 = -1$ is any scalar multiple of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 2$:

$x + 3z = 2x$, which rearranges to $3z = x$

$-y + 9z = 2y$, which rearranges to $9z = 3y$, so $3z = y$, which implies that $3z = x = y$, so $x = y$

$x - z = 2z$, which rearranges to $x = 3z$

So, an eigenvector of S corresponding to $\lambda_2 = 2$ is any scalar multiple of $\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$.

For $\lambda_3 = -2$:

$x + 3z = -2x$, which rearranges to $3z = -3x$, so $z = -x$

$-y + 9z = -2y$, which rearranges to $9z = -y$

$x - z = -2z$, which rearranges to $x = -z$

So, an eigenvector of S corresponding to $\lambda_3 = -2$ is any scalar multiple of $\begin{pmatrix} -1 \\ -9 \\ 1 \end{pmatrix}$.

e. $Tx = \lambda x$:

$$\begin{pmatrix}1 & 0 & 0 & 2\\1 & 3 & 0 & 0\\0 & 2 & -1 & 2\\0 & 0 & 0 & 3\end{pmatrix}\begin{pmatrix}w\\x\\y\\z\end{pmatrix} = \lambda\begin{pmatrix}w\\x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}w+2z\\w+3x\\2x-y+2z\\3z\end{pmatrix} = \begin{pmatrix}\lambda w\\\lambda x\\\lambda y\\\lambda z\end{pmatrix}$$

For $\lambda_1 = 1$:

$w + 2z = w$, which implies that $z = 0$

$w + 3x = x$, which rearranges to $w = -2x$

$2x - y + 2z = y$, which (with $z = 0$) rearranges to $x = y$

$3z = z$, which is already solved by $z = 0$

So, an eigenvector of T corresponding to $\lambda_1 = 1$ is any scalar multiple of $\begin{pmatrix}-2\\1\\1\\0\end{pmatrix}$.

For $\lambda_2 = -1$:

$w + 2z = -w$, which rearranges to $2z = -2w$, so $z = -w$

$w + 3x = -x$, which rearranges to $w = -4x$

$2x - y + 2z = -y$, which rearranges to $2x + 2z = 0$, so $z = -x = -w$

$3z = -z$, which implies that $z = 0$, which further implies that $z = x = w = 0$

This leaves $y = y$, so an eigenvector of T corresponding to $\lambda_2 = -1$ is any scalar multiple of $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$.

Finally, for $\lambda_3 = 3$:

$w + 2z = 3w$, which rearranges to $z = w$

$w + 3x = 3x$, so $w = z = 0$, and $x = x$

$2x - y + 2z = 3y$, which (with $w = z = 0$) rearranges to $x = 2y$

$3z = 3z$, which is already solved by $z = 0$.

So an eigenvector of T corresponding to $\lambda_3 = 3$ is any scalar multiple of $\begin{pmatrix}0\\2\\1\\0\end{pmatrix}$.

Support: Study Development offers workshops, short courses, 1 to 1 and small group tutorials.

- Book a tutorial or join a workshop on the Study Development tutorial and workshop webpage or search 'YSJ study development tutorials.'
- Access our Study Success resources on the Study Success webpage or search 'YSJ study success.'