Finding a basis for $R^{n}$

Study Development Quickguide

## What is a basis?

(Informally) a basis of $R^{n}$ is a set of vectors in $R^{n}$ that you can make any vector in $R^{n}$ out of by finding a linear combination of the basis vectors.

## Checking if a set of vectors is a basis

When we are given a set of vectors and asked ‘is this set of vectors a basis in $S$?’ we are being asked to check two things:

* Are the vectors linearly independent?
* Do they span $S$?

If the answer to both is yes, then we have a basis for $S$.

## Linear independence

A set of vectors being linearly independent means that we cannot form one of the vectors as a linear combination of the others. A linear combination of a set of vectors $v\_{1},v\_{2},…,v\_{n}$ is $a\_{1}v\_{1}+a\_{2}v\_{2}+…+a\_{n}v\_{n}$ where $a\_{1},a\_{2},…,a\_{n}$ are scalars.

For example,

$\left(\begin{matrix}1\\0\\1\end{matrix}\right)$, $\left(\begin{matrix}0\\2\\1\end{matrix}\right)$ and $\left(\begin{matrix}3\\-2\\2\end{matrix}\right)$ are not linearly independent, since $\left(\begin{matrix}0\\2\\1\end{matrix}\right)$= $3\left(\begin{matrix}1\\0\\1\end{matrix}\right) - \left(\begin{matrix}3\\-2\\2\end{matrix}\right)$.$ $

We can test for linear independence in a couple of different ways:

1. A set of vectors $v\_{1},v\_{2},…,v\_{n}$ are linearly independent if the only solution to $a\_{1}v\_{1}+a\_{2}v\_{2}+…+a\_{n}v\_{n}=0$ is $a\_{1}=a\_{2}=…=a\_{n}=0$.
2. The determinant of a matrix is 0 if and only if the columns are linearly dependent.

To use this method, we write the vectors as the columns of a matrix and then find the determinant.

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## Spanning

When we say a set of vectors $v\_{1},v\_{2},…,v\_{n}$ span $S$, we mean that we can find any vector $v$ in $S$ by writing it as $a\_{1}v\_{1}+a\_{2}v\_{2}+…+a\_{n}v\_{n}=v$ for scalars $a\_{1},a\_{2},…,a\_{n}$.

There is a very useful result that ‘$n$ linearly independent vectors will span $R^{n}$’ so, if you are testing whether your vectors span $R^{n}$, and you have $n$ linearly independent vectors, you know that they span $R^{n}$.

## Showing we have a basis in one step

We may rewrite $a\_{1}v\_{1}+a\_{2}v\_{2}+…+a\_{n}v\_{n}=v$ as $\left(\begin{matrix}a\_{1}&a\_{2}&…&a\_{n}\end{matrix}\right)V=v$, where $V$ is a matrix whose columns are $v\_{1},v\_{2},…,v\_{n}$. If $V$ is invertible, then $v\_{1},v\_{2},…,v\_{n}$ are a basis for $R^{n}$. The explanation of this is in the factsheet.

We also know that a matrix is invertible if and only if its determinant is not 0.

Therefore, to show that we have a basis:

Write the vectors as the columns of a matrix, find the determinant. If the determinant is 0, we do not have a basis. If the determinant is not 0, we have a basis.

## Finding a basis

Now that we know how to check if vectors are a basis, finding a basis is a pretty simple task. We know that we need to find the columns of an nxn matrix such that its determinant isn’t zero.

For example, if we are asked to find a basis for $R^{3}$ we can simply fill in the blanks of a 3x3 matrix. Start by choosing any vector in $R^{3}$. We’ll call this $v\_{1}=\left(\begin{matrix}v\_{1,1}\\v\_{1,2}\\v\_{1,3}\end{matrix}\right)$ Then, choose a second vector in $R^{3}$- as long as this isn’t a scalar multiple of your first vector it will work. We write this as $v\_{2}=\left(\begin{matrix}v\_{2,1}\\v\_{2,2}\\v\_{2,3}\end{matrix}\right)$. Next, we write

$$\left(\begin{matrix}v\_{1,1}&v\_{2,1}&v\_{3,1}\\v\_{1,2}&v\_{2,2}&v\_{3,2}\\v\_{1,3}&v\_{2,3}&v\_{3,3}\end{matrix}\right) $$

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We then find the determinant and choose values for $v\_{3,1}$, $v\_{3,2}$ and $v\_{3,3}$ such that the determinant is not 0.

If we are finding a basis for $R^{n}$ with $n>3$, we have to check for linear independence each time we add a new vector past the second vector added.

**Hint:** there are vectors that we sometimes call the ‘standard basis vectors’. These are vectors that have only one non-zero entry that is 1. These are usually the easiest way to find a basis. For example, for $R^{3}$, the ‘standard basis vectors’ are $\left(\begin{matrix}1\\0\\0\end{matrix}\right)$, $\left(\begin{matrix}0\\1\\0\end{matrix}\right)$ and $\left(\begin{matrix}0\\0\\1\end{matrix}\right)$. If you are unsure, try selecting the first two basis vectors in this format, and it should make it much easier to find a third vector.

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