Finding a basis for R^n

Study Development Quickguide

What is a basis?

(Informally) a basis of R^n is a set of vectors in R^n that you can make any vector in R^n out of by finding a linear combination of the basis vectors.

Checking if a set of vectors is a basis

When we are given a set of vectors and asked 'is this set of vectors a basis in S?' we are being asked to check two things:

- Are the vectors linearly independent?
- Do they span S?

If the answer to both is yes, then we have a basis for S.

Linear independence

A set of vectors being linearly independent means that we cannot form one of the vectors as a linear combination of the others. A linear combination of a set of vectors $v_1, v_2, \dots, v_n$ is $a_1v_1 + a_2v_2 + \cdots + a_nv_n$ where $a_1, a_2, \dots, a_n$ are scalars.

For example,

$\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\begin{pmatrix}0\\2\\1\end{pmatrix}$ and $\begin{pmatrix}3\\-2\\2\end{pmatrix}$ are not linearly independent, since $\begin{pmatrix}0\\2\\1\end{pmatrix} = 3\begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}3\\-2\\2\end{pmatrix}$.

We can test for linear independence in a couple of different ways:

1. A set of vectors $v_1, v_2, \dots, v_n$ are linearly independent if the only solution to $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ is $a_1 = a_2 = \cdots = a_n = 0$.
2. The determinant of a matrix is 0 if and only if the columns are linearly dependent.

To use this method, we write the vectors as the columns of a matrix and then find the determinant.

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Spanning

When we say a set of vectors $v_1, v_2, \dots, v_n$ span S, we mean that we can find any vector v in S by writing it as $a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$ for scalars $a_1, a_2, \dots, a_n$.

There is a very useful result that 'n linearly independent vectors will span R^n' so, if you are testing whether your vectors span R^n, and you have n linearly independent vectors, you know that they span R^n.

Showing we have a basis in one step

We may rewrite $a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$ as $(a_1 \quad a_2 \quad \dots \quad a_n)V = v$, where V is a matrix whose columns are $v_1, v_2, \dots, v_n$. If V is invertible, then $v_1, v_2, \dots, v_n$ are a basis for R^n. The explanation of this is in the factsheet.

We also know that a matrix is invertible if and only if its determinant is not 0.

Therefore, to show that we have a basis:

Write the vectors as the columns of a matrix, find the determinant. If the determinant is 0, we do not have a basis. If the determinant is not 0, we have a basis.

Finding a basis

Now that we know how to check if vectors are a basis, finding a basis is a pretty simple task. We know that we need to find the columns of an nxn matrix such that its determinant isn't zero.

For example, if we are asked to find a basis for R^3 we can simply fill in the blanks of a 3x3 matrix.

Start by choosing any vector in R^3. We'll call this $v_1 = \begin{pmatrix} v_{1,1} \\ v_{1,2} \\ v_{1,3} \end{pmatrix}$ Then, choose a second vector in R^3- as long as this isn't a scalar multiple of your first vector it will work. We write this as $v_2 = \begin{pmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{pmatrix}$.

Next, we write

$$\begin{pmatrix} v_{1,1} & v_{2,1} & v_{3,1} \\ v_{1,2} & v_{2,2} & v_{3,2} \\ v_{1,3} & v_{2,3} & v_{3,3} \end{pmatrix}$$

We then find the determinant and choose values for $v_{3,1}$, $v_{3,2}$ and $v_{3,3}$ such that the determinant is not 0.

If we are finding a basis for R^n with $n > 3$, we have to check for linear independence each time we add a new vector past the second vector added.

Hint: there are vectors that we sometimes call the 'standard basis vectors'. These are vectors that have only one non-zero entry that is 1. These are usually the easiest way to find a basis. For example, for R^3, the 'standard basis vectors' are $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$. If you are unsure, try selecting the first two basis vectors in this format, and it should make it much easier to find a third vector.

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