Finding a basis for

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## What is a basis?

(Informally) a basis of is a set of vectors in that you can make any vector in out of by finding a linear combination of the basis vectors. For example,

and are a basis for because we can write any vector in as a linear combination of the vectors, as .

## Why find a basis?

As with most things we do in linear algebra, finding a basis is a great way to condense a lot of information into a much smaller package. Imagine listing out every vector in , versus simply saying 'every vector in can be written as a linear combination of , and ’.

## Checking if a set of vectors is a basis

When we are given a set of vectors and asked ‘is this set of vectors a basis in ?’ we are being asked to check two things:

* Are the vectors linearly independent?
* Do they span ?

If the answer to both is yes, then we have a basis for .

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## Linear independence

A set of vectors being linearly independent means that we cannot form one of the vectors as a linear combination of the others. A linear combination of a set of vectors  is where are scalars.

For example,

, and are not linearly independent, since = .

We can test for linear independence in a couple of different ways:

1. A set of vectors are linearly independent if the only solution to  is .

For example, for the vectors and , we find

(for scalars and ). Multiplying out, we get

and so, we have

Solving these simultaneously gives us and so, . Giving , and so . Therefore, the only solution we have is , so the vectors are linearly independent.

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1. The determinant of a matrix is 0 if and only if the columns are linearly dependent.

To use this method, we write the vectors as the columns of a matrix and then find the determinant.

For example, to test if , and are linearly independent, we find:

Therefore, the vectors are linearly independent.

## Spanning

When we say a set of vectors span , we mean that we can find any vector in by writing it as for scalars .

There is a very useful result that ‘ linearly independent vectors will span ’ (proof: see below in the method for showing independence and span in one step).

So, if you are testing whether your vectors span , and you have linearly independent vectors, we know that they span .

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## Showing we have a basis in one step

We may rewrite as , where is a matrix whose columns are . If is invertible, we have

Therefore, if exists, we know that there is always a solution for , and so is spanned by

**Note:**  if is invertible, this also means are linearly independent, so we have also shown that is a basis in this one calculation.

We also know that a matrix is invertible if and only if its determinant is not 0.

Therefore, to show that we have a basis:

Write the vectors as the columns of a matrix, find the determinant. If the determinant is 0, we do not have a basis. If the determinant is not 0, we have a basis.

For example, to test if the vectors , and are a basis of , we perform the following calculation:

Therefore, they are a basis of .

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## Finding a basis

Now that we know how to check if vectors are a basis, finding a basis is a pretty simple task. We know that we need to find the columns of an nxn matrix such that its determinant isn’t zero.

For example, if we are asked to find a basis for we can simply fill in the blanks of a 3x3 matrix. Start by choosing any vector in . We’ll call this Then, choose a second vector in - as long as this isn’t a scalar multiple of your first vector it will work. We write this as Next, we write

We then find the determinant and choose values for , and such that the determinant is not 0.

If we are finding a basis for with , we have to check for linear independence each time we add a new vector past the second vector added.

**Hint:** there are vectors that we sometimes call the ‘standard basis vectors’. These are vectors that have only one non-zero entry that is 1. These are usually the easiest way to find a basis. For example, for , the ‘standard basis vectors’ are , and . If you are unsure, try selecting the first two basis vectors in this format, and it should make it much easier to find a third vector.

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For example, find a basis for :

First, we choose as one of the vectors. Since is not a scalar multiple of our first vector, we choose this as our second vector. We select as a potential third vector, but we must first check it that these three vectors are linearly independent:

gives , so we have as the only solution, and so they are linearly independent.

Finally, we set up:

We find the determinant:

Therefore, we have that , and can be any values in that we’d like, and can be anything in except 0. So, we could have a basis of .

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