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Finding a basis for R^n

Study Development Factsheet

What is a basis?

(Informally) a basis of R^n is a set of vectors in R^n that you can make any vector in R^n out of by finding a linear combination of the basis vectors. For example,

$\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are a basis for R^2 because we can write any vector in R^2 as a linear combination of the vectors, as $\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$.

Why find a basis?

As with most things we do in linear algebra, finding a basis is a great way to condense a lot of information into a much smaller package. Imagine listing out every vector in R^3, versus simply saying 'every vector in R^3 can be written as a linear combination of $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$'.

Checking if a set of vectors is a basis

When we are given a set of vectors and asked 'is this set of vectors a basis in S?' we are being asked to check two things:

- Are the vectors linearly independent?
- Do they span S?

If the answer to both is yes, then we have a basis for S.

Linear independence

A set of vectors being linearly independent means that we cannot form one of the vectors as a linear combination of the others. A linear combination of a set of vectors $v_1, v_2, \dots, v_n$ is $a_1v_1 + a_2v_2 + \cdots + a_nv_n$ where $a_1, a_2, \dots, a_n$ are scalars.

For example,

$\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\begin{pmatrix}0\\2\\1\end{pmatrix}$ and $\begin{pmatrix}3\\-2\\2\end{pmatrix}$ are not linearly independent, since $\begin{pmatrix}0\\2\\1\end{pmatrix} = 3\begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}3\\-2\\2\end{pmatrix}$.

We can test for linear independence in a couple of different ways:

1. A set of vectors $v_1, v_2, \dots, v_n$ are linearly independent if the only solution to $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ is $a_1 = a_2 = \cdots = a_n = 0$.

For example, for the vectors $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\1\end{pmatrix}$, we find

$$a\begin{pmatrix}1\\2\end{pmatrix} + b\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

(for scalars a and b). Multiplying out, we get

$$\begin{pmatrix}a\\2a\end{pmatrix} + \begin{pmatrix}3b\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

and so, we have

$$(1)\ a + 3b = 0$$
$$(2)\ 2a + b = 0$$

Solving these simultaneously gives us $(2) - (1)$: $-5b = 0$ and so, $b = 0$. Giving (1): $a + 0 = 0$, and so $a = 0$. Therefore, the only solution we have is $a = b = 0$, so the vectors are linearly independent.

2. The determinant of a matrix is 0 if and only if the columns are linearly dependent.

To use this method, we write the vectors as the columns of a matrix and then find the determinant.

For example, to test if $\begin{pmatrix}1\\0\\2\end{pmatrix}$, $\begin{pmatrix}3\\3\\4\end{pmatrix}$ and $\begin{pmatrix}0\\0\\2\end{pmatrix}$ are linearly independent, we find:

$$det\begin{pmatrix}1 & 3 & 0\\0 & 3 & 0\\2 & 4 & 2\end{pmatrix} = 6 \neq 0$$

Therefore, the vectors are linearly independent.

Spanning

When we say a set of vectors $v_1, v_2, \ldots, v_n$ span S, we mean that we can find any vector v in S by writing it as $a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$ for scalars $a_1, a_2, \ldots, a_n$.

There is a very useful result that 'n linearly independent vectors will span R^n' (proof: see below in the method for showing independence and span in one step).

So, if you are testing whether your vectors span R^n, and you have n linearly independent vectors, we know that they span R^n.

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Showing we have a basis in one step

We may rewrite $a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$ as $(a_1 \quad a_2 \quad \ldots \quad a_n)V = v$, where V is a matrix whose columns are $v_1, v_2, \ldots, v_n$. If V is invertible, we have

$$(a_1 \quad a_2 \quad \ldots \quad a_n)V = v$$

$$(a_1 \quad a_2 \quad \ldots \quad a_n)VV^{-1} = vV^{-1}$$

$$(a_1 \quad a_2 \quad \ldots \quad a_n)I_n = vV^{-1}$$

$$(a_1 \quad a_2 \quad \ldots \quad a_n) = vV^{-1}$$

Therefore, if V^{-1} exists, we know that there is always a solution for $a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$, and so S is spanned by $v_1, v_2, \ldots, v_n$.

Note: if V is invertible, this also means $v_1, v_2, \ldots, v_n$ are linearly independent, so we have also shown that $v_1, v_2, \ldots, v_n$ is a basis in this one calculation.

We also know that a matrix is invertible if and only if its determinant is not 0.

Therefore, to show that we have a basis:

Write the vectors as the columns of a matrix, find the determinant. If the determinant is 0, we do not have a basis. If the determinant is not 0, we have a basis.

For example, to test if the vectors $\begin{pmatrix}1\\4\\1\end{pmatrix}$, $\begin{pmatrix}0\\2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\3\\0\end{pmatrix}$ are a basis of R^3, we perform the following calculation:

$$det\begin{pmatrix}1 & 0 & 1\\4 & 2 & 3\\1 & 1 & 0\end{pmatrix} = -1 \neq 0$$

Therefore, they are a basis of R^3.

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Finding a basis

Now that we know how to check if vectors are a basis, finding a basis is a pretty simple task. We know that we need to find the columns of an nxn matrix such that its determinant isn't zero.

For example, if we are asked to find a basis for R^3 we can simply fill in the blanks of a 3x3 matrix.

Start by choosing any vector in R^3. We'll call this $v_1 = \begin{pmatrix} v_{1,1} \\ v_{1,2} \\ v_{1,3} \end{pmatrix}$ Then, choose a second vector in R^3- as long as this isn't a scalar multiple of your first vector it will work. We write this as $v_2 = \begin{pmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{pmatrix}$Next, we write

$$\begin{pmatrix} v_{1,1} & v_{2,1} & v_{3,1} \\ v_{1,2} & v_{2,2} & v_{3,2} \\ v_{1,3} & v_{2,3} & v_{3,3} \end{pmatrix}$$

We then find the determinant and choose values for $v_{3,1}$, $v_{3,2}$ and $v_{3,3}$ such that the determinant is not 0.

If we are finding a basis for R^n with $n > 3$, we have to check for linear independence each time we add a new vector past the second vector added.

Hint: there are vectors that we sometimes call the 'standard basis vectors'. These are vectors that have only one non-zero entry that is 1. These are usually the easiest way to find a basis. For example, for R^3, the 'standard basis vectors' are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. If you are unsure, try selecting the first two basis vectors in this format, and it should make it much easier to find a third vector.

For example, find a basis for R^4:

First, we choose $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ as one of the vectors. Since $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ is not a scalar multiple of our first vector, we choose this as our second vector. We select $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ as a potential third vector, but we must first check it that these three vectors are linearly independent:

$a\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ gives $\begin{pmatrix}a\\b\\c\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$, so we have $a=b=c=0$ as the only solution, and so they are linearly independent.

Finally, we set up:

$$\begin{pmatrix}1 & 0 & 0 & v_{4,1}\\0 & 1 & 0 & v_{4,2}\\0 & 0 & 1 & v_{4,3}\\0 & 0 & 0 & v_{4,4}\end{pmatrix}$$

We find the determinant:

$$det\begin{pmatrix}1 & 0 & 0 & v_{4,1}\\0 & 1 & 0 & v_{4,2}\\0 & 0 & 1 & v_{4,3}\\0 & 0 & 0 & v_{4,4}\end{pmatrix}=1\left(1(v_{4,4})\right)=v_{4,4}$$

Therefore, we have that $v_{4,1}$, $v_{4,2}$ and $v_{4,3}$ can be any values in R that we'd like, and $v_{4,4}$ can be anything in R except 0. So, we could have a basis of $\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}2\\3\\0\\1\end{pmatrix}$.

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