Vectors

Study Development Factsheet

Notation

R^n: a vector with n real entries

C^n: a vector with n complex entries

$R^{n\times 1}$: a vertical vector with n real entries

$R^{1\times n}$: a horizontal vector with n real entries

$\vec{x}$: a vector x (any letter can be used)

v: a vector v (any letter can be used- usually u, v and w)

We write a vertical vector with n entries in the form $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$. Square brackets, such as $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, mean the same as the curved bracket notation. The same vectors can be written horizontally, as $(v_1\ v_2\ \dots\ v_n)$ or $[v_1\ v_2\ \dots\ v_n]$. We choose whether to write a vector horizontally or vertically depending on what we intend to use it for, but they both represent the same vector.

$v = 0$: this means the 'zero vector', which is a vector with n entries that are all zero. Note that $0u = u0 = 0$ for any vector u.

Operations

Addition and subtraction

We add vectors by adding entries in the same position to each other. For example:

$$\begin{pmatrix}1\\0\\4\end{pmatrix} + \begin{pmatrix}2\\5\\8\end{pmatrix} = \begin{pmatrix}1+2\\0+5\\4+8\end{pmatrix} = \begin{pmatrix}3\\5\\12\end{pmatrix}$$

Subtraction works similarly, for example:

$$\begin{pmatrix}10\\4\\6\end{pmatrix} - \begin{pmatrix}3\\2\\8\end{pmatrix} = \begin{pmatrix}10-3\\4-2\\6-8\end{pmatrix} = \begin{pmatrix}7\\2\\-2\end{pmatrix}$$

This is a good opportunity to see how, in this example, it is more appropriate to use vertical vectors, as the horizontal version looks like this:

$$(10\ 4\ 6) - (3\ 2\ 8) = (10-3\ \ 4-2\ \ 6-8) = (7\ \ 2\ \ -2)$$

Like when we add numbers, we have that $u + -v = u - v$ for vectors u and v.

Scalar multiplication

When we multiply be a scalar (a number or variable that is not a vector) we simply multiply each entry in the vector by the scalar. For example:

$$3\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\times 2\\3\times 1\end{pmatrix} = \begin{pmatrix}6\\3\end{pmatrix}$$

Norm

The norm of a vector, $||v||$, represents the length of the vector. It is calculated with the formula:

$$\left\|\begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix}\right\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

For example: $\left\|\begin{pmatrix}1\\3\\2\end{pmatrix}\right\| = \sqrt{1^2+3^2+2^2} = \sqrt{1+9+4} = \sqrt{14}$

Note: though, mathematically, the answer to $||v||$ could be negative, this doesn't make sense when we say that the norm represents the length of v, so we use the positive root.

Dot product

The dot product of two vectors, also called the inner product or scalar product, tells you (informally) the amount that one vector goes in the same direction as the other vector. If the vectors go in perpendicular directions (i.e. not at all in the same direction) their dot product is zero, and this is called orthogonality. If the vectors go in exactly the same direction, the dot product is equal to the length of the vector squared.

The formula for the inner product is given by:

$$\langle u, v\rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

For example:

$$\left\langle \begin{pmatrix} 10 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix} \right\rangle = (10 \times 2) + (0 \times 8) + (4 \times 3) = 20 + 24 = 44$$

Cross product

The cross product of two vectors finds a vector that is perpendicular to both of them. The formula to find the cross product between $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is:

$$\mathrm{u} \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

For example,

$$\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 10 \\ 12 \end{pmatrix} = \begin{pmatrix} (0 \times 12) - (4 \times 10) \\ (4 \times 1) - (2 \times 12) \\ (2 \times 10) - (0 \times 1) \end{pmatrix} = \begin{pmatrix} -40 \\ -20 \\ 20 \end{pmatrix}$$

The cross product is also given by

$$u \times v = ||u||\,||v||\, sin(\theta)\, n$$

Where θ is the angle between u and v and n is a unit vector perpendicular to u and v.

Note: The cross product only exists (non-trivially) in 3- and 7-dimensional space. The formula for the 7-dimensional cross product can be found online.

Angle between two vectors

The formula for finding the angle θ between two vectors u and v is as follows:

$$\theta = cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right)$$

For example, the angle between $\binom{2}{0}$ and $\binom{\sqrt{3}}{1}$ is given by:

$$\theta = cos^{-1}\left(\frac{\binom{2}{0}\cdot\binom{\sqrt{3}}{1}}{\left|\binom{2}{0}\right|\left|\binom{\sqrt{3}}{1}\right|}\right) = cos^{-1}\left(\frac{2\sqrt{3}}{\sqrt{4}+\sqrt{4}}\right) = cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30° \text{ (or } \frac{\pi}{6} \text{ radians).}$$

Orthogonality

If two vectors are perpendicular to each other, we call them orthogonal. If they are both of length 1, we call them orthonormal.

To test if they are orthogonal, we find their dot product. If this is 0, they are orthogonal, as when we calculate the angle between them, θ, we have $\theta = cos^{-1}(0) = 90°$ (or $\frac{\pi}{2}$ radians).

Projection onto a vector

When we project a vector u onto a vector v, we draw a line perpendicular to vector v that crosses through u, and then we find the point on v that this line crosses through. This is answering the (informal) question: how much does one vector (u) go in the direction of another vector (v)?

The formula for calculating the projection of u onto v ($\text{proj}_v(u)$) is as follows:

$$\text{proj}_v(u) = \frac{u \cdot v}{v \cdot v} v$$

For example, the projection of $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ onto $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ is given by:

$$\text{proj}_{(131)}\left(\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}\right) = \frac{\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \frac{10}{11} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{10}{11} \\ \frac{30}{11} \\ \frac{10}{11} \end{pmatrix}.$$

Note that the vector we end up with is a multiple of v, so will go in the direction of v (or $-v$).

Support: Study Development offers workshops, short courses, 1 to 1 and small group tutorials.

- Join a tutorial or workshop on the Study Development tutorial and workshop webpage or search 'YSJ study development tutorials.'
- Access our Study Success resources on the Study Success webpage or search 'YSJ study success.'