# The Law of Total Probability

Study Development Factsheet

The (discrete) Law of Total Probability (also called the Law of Alternatives) states that if events $\{B_n : n = 1, 2, \ldots\}$ are pairwise dijoint and their union covers the entire sample space, then

$$P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n)P(B_n)$$

(provided each $B_n$ is measurable and $A$ exists in the same probability space).

So, what does any of that mean? We'll break it down:

- Discrete: separate and distinct events. For example, the results of a dice roll are discrete, and the age of a tree is discrete. This is because we cannot subdivide these categories- for example, we can't get 3.5 on a dice roll. There is a fixed, non-infinite number of possible outcomes.
  Something like the volume of water leaking from a ceiling would not be discrete, since it would take us literally forever to write out every possible volume of water we could measure. This is an example of a continuous variable.

- Events $\{B_n : n = 1, 2, \ldots\}$ : this is a set of possible outcomes. If we think of the outcomes of rolling a 6-sided dice as the events $\{B_n : n = 1, 2, \ldots\}$, then we have $B_1 = 1$, $B_2 = 2$, $B_3 = 3$, $B_4 = 4$, $B_5 = 5$, $B_6 = 6$.

- Pairwise disjoint: there is no possible way to have two different events in $\{B_n : n = 1, 2, \ldots\}$ happen at the same time. For example, if $\{B_n : n = 1, 2\}$ are the outcomes of tossing a coin one time, then we have $B_1 =$ heads and $B_2 =$ tails, so we cannot have $B_1$ and $B_2$ at the same time.

- Their union covers the entire sample space: $\{B_n : n = 1, 2, \ldots\}$ covers every possibility. If we were looking at the 6-sided dice example, then we need to have an event $B_n$ for each face of the dice. If we had, for example, $B_1 = 2$, $B_2 = 4$ and $B_3 = 6$ as the events $\{B_n : n = 1, 2, 3\}$ then their union does not cover the entire sample space.

- $B_n$ is measurable: don't worry too much about this. In mathematics, the concept of a nonmeasurable event is very abstract and requires a lot of further knowledge to define. Unless you have been told that an event $B_n$ is nonmeasurable, in which case you cannot use the law of total probability, assume they are measurable.

- $A$ exists in the same probability space: if we are, for example, talking about plants, and $\{B_n : n = 1, 2, \ldots\}$ is the different colours that the flowers can be, then $A$ could be another plant property, such as the number of leaves the plant has.

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**Example**

*A gardener is planting three different kinds of bulbs. They have 25 large bulbs, 25 medium, and 25 small bulbs. The probability that a large bulb will produce a plant with red flowers is 0.75, the probability that a medium bulb will produce a plant with red flowers is 0.6, and the probability that the small bulbs will is 0.45. The gardener chooses a bulb at random and plants one. What is the probability that the plant it produces will have red flowers? You may assume that the bulbs will always grow into a plant that flowers.*

This is a huge chunk of information. We start by checking that we can actually use the law of total probability:

A bulb cannot be two different sizes at once, so the events $B_1$ = large bulb is chosen, $B_2$ = medium bulb is chosen, and $B_3$ = small bulb is chosen are pairwise disjoint.

The gardener choses a bag of bulbs at random from the three bags, so $\{B_n : n = 1, 2, 3\}$ covers the entire sample space.

$\{B_n : n = 1, 2, 3\}$ are discrete outcomes (for example, we can't have a bulb that is 'medium and a half').

Therefore, we can use the law of total probability.

We now put the information into a nicer format. We say that $A$ is the event that the plant has red flowers, $B_1$ is the bulb being large, $B_2$ is the bulb being medium, and $B_3$ is the bulb being small.

We have:

$$P(B_1) = \frac{25}{75} = \frac{1}{3}$$

$$P(B_2) = \frac{1}{3}$$

$$P(B_3) = \frac{1}{3}$$

We get this since there are 25 bulbs in each category, and 75 bulbs total, so each type of bulb has a probability of $\frac{25}{75}$ of being chosen at random.

We also have (from the information given):

$$P(A|B_1) = 0.75$$

$$P(A|B_2) = 0.6$$

$$P(A|B_3) = 0.45$$

We can now find $P(A)$ using the law of total probability:

$$P(A) = \sum_n P(A|B_n)P(B_n) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$

$$= 0.75 \times \frac{1}{3} + 0.6 \times \frac{1}{3} + 0.45 \times \frac{1}{3} = 0.6$$

So, the probability that a bulb will produce a plant with red flowers is 0.6.

Hopefully this has shown that, although the law itself looks quite complicated, actually using it isn't too confusing.

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