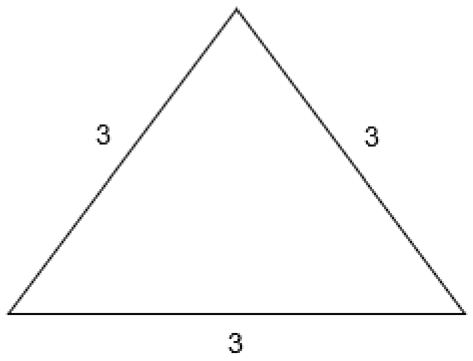
Trigonometry

Study Development Worksheet

Example

What kind of triangle is this?



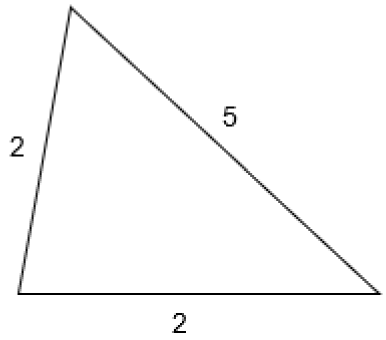
Answer

Since the three sides are all the same length, this is an equilateral triangle.

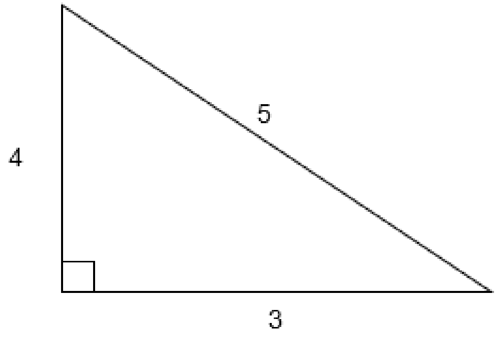
Questions

Define, with reasons, the following triangles:

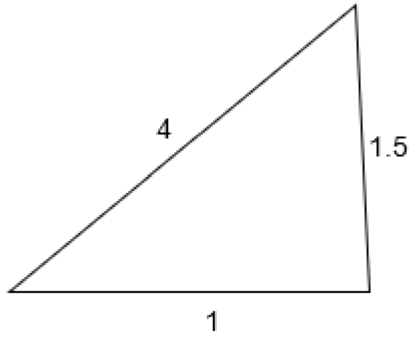
a)



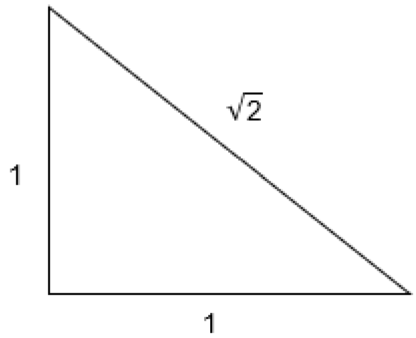
b)



c)



d)



Answers

a) Since there are only two sides of the same length, this is an isosceles triangle.

b) Due to the fact that the side lengths form a Pythagorean triple (and this has been shown by the square in the right-angle vertex), this is a right-angle triangle. Since the side lengths are all different, this is also a scalene triangle.

c) Since the side lengths are all different, this is a scalene triangle.

d) Due the fact that the side lengths follow Pythagoras's theorem, this is a right-angle triangle. Since two of the side lengths are the same lengths, this is also an isosceles triangle.

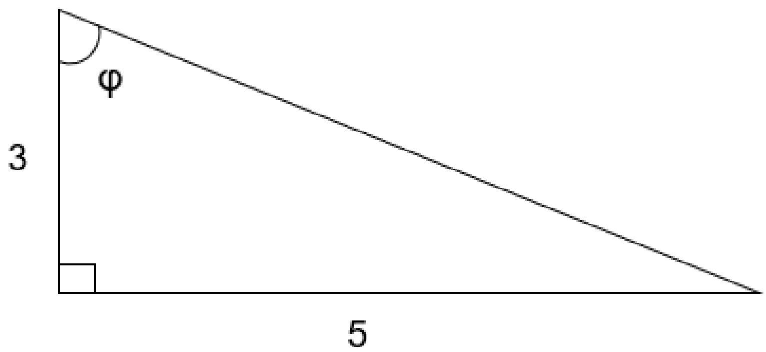
Right-Angle Triangles

Study Development Worksheet

Example

A right-angled triangle has two shorter side lengths of 3 and 5. The angle between the shortest side length and the hypotenuse is labelled as φ.

Find the angle φ and the length of the hypotenuse.



Answer

Using Pythagoras' Theorem ($a^2 + b^2 = c^2$), we have that the hypotenuse, c, is equal to the square root of $a^2 + b^2$.

$a^2 + b^2 = 3^2 + 5^2 = 9 + 25 = 36 = c^2$

$c = \sqrt{36} = 6$

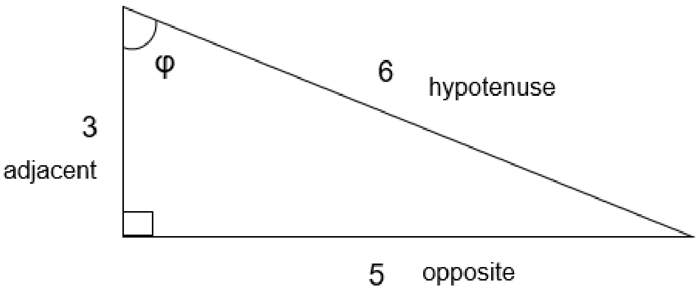
(We know that potentially $c = \sqrt{36} = -6$, however since it is representing a length, we must only use the positive value for c, since lengths are always positive.)

Note: When all 3 side lengths of a right-angled triangle are whole numbers, we call them a Pythagorean triple. Examples include {3, 5 and 6}, {3, 4 and 5} and {5, 12 and 13}.

We calculate the angle by using one of the sine, cosine or tangent ratios. We may use any here, as we have all 3 side lengths. We label the sides: the side of length 3 is adjacent to the angle φ, the side length of 5 is opposite the angle φ, and the side length of 6 is the hypotenuse.

Right-Angle Triangles

Study Development Worksheet



We then use the tangent rule:

$$\tan(\varphi) = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{3}$$

Therefore, $\varphi = \tan^{-1}\left(\frac{5}{3}\right) = 59.036°$.

Right-Angle Triangles

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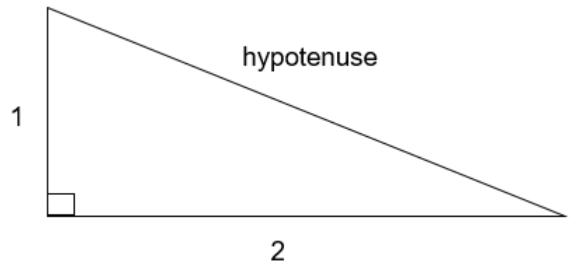
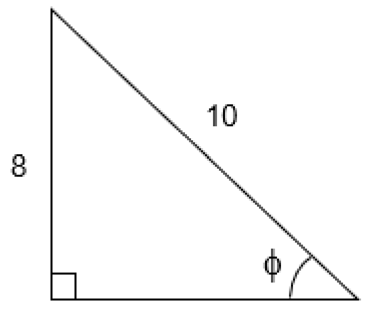
Questions

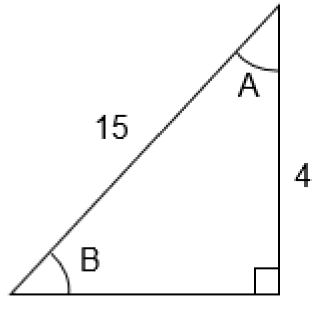
1. Find the length of the hypotenuse on a right-angled triangle with shorter side lengths 1 and 2.



2. A right-angled triangle has a hypotenuse of length 10, and a shorter side length of length 8. The angle opposite the side of length 8 is labelled as ϕ. Find the length of the other shorter side and the size of the angle ϕ.



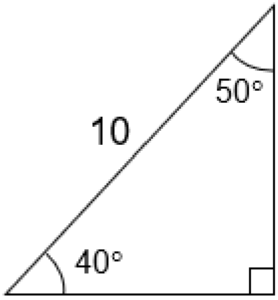
3. A right-angled triangle has a hypotenuse of length 15 and a shorter side that is length 4. What are the sizes of the internal angles in degrees?



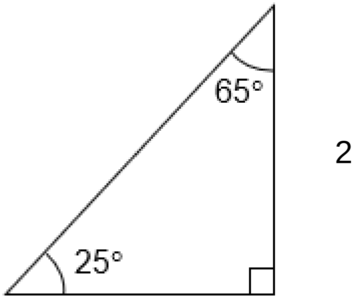
4. A right-angled triangle has a hypotenuse of length 10. The internal angles are 40°, 50° and 90°. What are the lengths of the shorter sides?

Right-Angle Triangles

Study Development Worksheet



5. A right-angled triangle has internal angles of size 65°, 25° and 90°. The side that is adjacent to the 65° angle is length 2. What are the lengths of the other sides?



Right-Angle Triangles

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Answers

1. Using Pythagoras' Theorem, we have that $a^2 + b^2 = c^2$. Therefore, $c^2 = 1^2 + 2^2 = 1 + 4 = 5$, and so the hypotenuse (c), is $c = \sqrt{5}$.
2. Using Pythagoras' Theorem, we have $a^2 + 8^2 = 10^2$, so $a^2 = 100 - 64 = 36$, and so $a = \sqrt{36} = 6$.

   We calculate the angle using the sine ratio: $\sin(\phi) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{8}{10}$, and so

   $\Phi = \sin^{-1}\left(\frac{8}{10}\right) = 53.13°$.
3. We begin by calculating the size of angle A. Since we have the length of the hypotenuse and the length of the side that is adjacent to the angle A, we use the cosine ratio:

   $\cos(A) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{15}$, and so, $A = \cos^{-1}\left(\frac{4}{15}\right) = 74.534°$.

   We then calculate the size of angle B. We could do this using the sine ratio, however, we know that the internal angles of a triangle sum to 180°, and so we just need to calculate $180° - 90° - 74.534° = 15.466°$.
4. We begin with the side that is opposite the angle that is 40°. Since we have the angle, the hypotenuse, and we want the opposite side, we use the sine ratio:

   $sin(40°) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{opposite}}{10}$. Therefore, the opposite side length is calculated by

   $10 \times \sin(40°) = 10 \times 0.643 = 6.43$.

   We calculate the other side length by using Pythagoras' Theorem. $a^2 + 6.43^2 = 10^2$, and so,

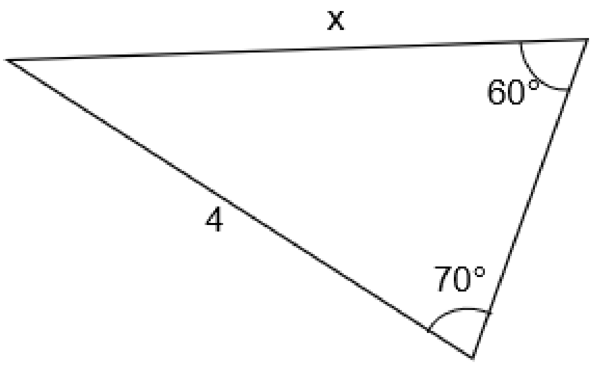
   $a = \sqrt{100 - 41.345} = \sqrt{68.6551} = 7.659$.
5. We begin by calculating the length of the hypotenuse:

   $\sin(25°) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{2}{\text{hypotenuse}}$, and so, hypotenuse $= \frac{2}{\sin(25°)} = \frac{2}{0.423} = 4.728$.

   We can now use Pythagoras' Theorem to calculate the final side length: $a^2 + 2^2 = 4.728^2$, and so, $a = \sqrt{4.728^2 - 2^2} = \sqrt{22.354 - 4} = \sqrt{18.354} = 4.284$.

Example

Find the length of side x.



Answer

We use the sine rule:

$$\frac{a}{sin(A)} = \frac{b}{sin(B)}$$

Plugging in the values that we know, we have:

$$\frac{x}{sin(70)} = \frac{4}{sin(60)}$$

We then multiply both sides by $sin(70)$:

$$x = \frac{4\,sin(70)}{sin(60)}$$

We compute this using a calculator, to get:

$x = 4.34.$

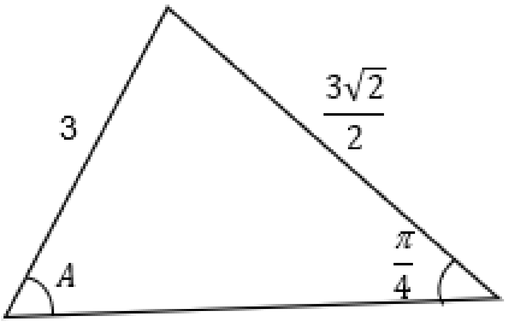
Sine Rule

Study Development Worksheet

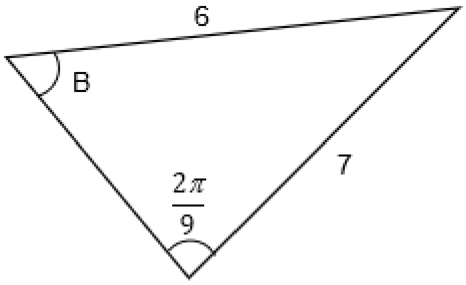
Questions

1. A triangle has two sides of the same length. The other side is length 5, and is opposite an angle of 120°. The other two angles are both 30°. What is the length of one of the sides that are the same length?

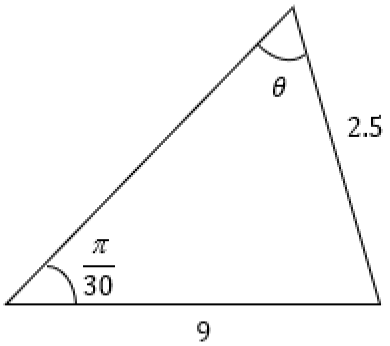
2. Find the size of the angle A (in radians):



3. Find the two possible degrees for the angle B in radians.



4. How many solutions are there for the angle θ? What are they?

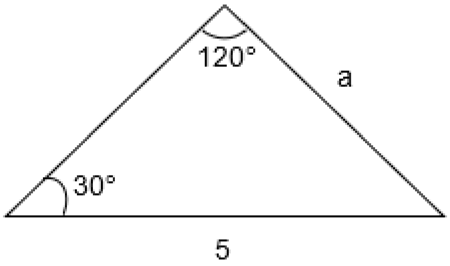


Sine Rule

Study Development Worksheet

Answers

1. We can draw a diagram:



We use the sine rule to find the side length: $\frac{a}{sin(30)} = \frac{5}{sin(120)}$, therefore $a = \frac{5\,sin(30)}{sin(120)}$,

which means that $a = 2.89$ (rounded to 2 decimal places).

2. Using the sine rule, we have that $\frac{\left(\frac{3\sqrt{2}}{2}\right)}{sin(A)} = \frac{3}{sin\left(\frac{\pi}{4}\right)}$, and then we rearrange to get $\frac{\left(\frac{3\sqrt{2}}{2}\right)sin\left(\frac{\pi}{4}\right)}{3} = sin(A)$. We can simplify this to get $\frac{\left(\frac{3\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{3} = \frac{1}{2} = sin(A)$. Finally, we find $\mathrm{A} = sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

3. Using the sine rule, $\frac{6}{sin\left(\frac{2\pi}{9}\right)} = \frac{7}{sin(B)}$, which gives us $sin(B) = \frac{7\,sin\left(\frac{2\pi}{9}\right)}{6}$. This gives us $B = sin^{-1}\left(\frac{7\,sin\left(\frac{2\pi}{9}\right)}{6}\right) = 0.848^c$. The other answer, due to the nature of the sine function, is given by $x_2 = \pi - x_1$ (where x_1 and x_2 are the two answers). Therefore, the answers are 0.848^c and $\pi^c - 0.848^c = 2.29^c$.

Note: There is only one possible angle size for question 2 because the other potential angle is $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$, and if we sum the two known angles we get $\frac{5\pi}{6} + \frac{\pi}{4} = \frac{13\pi}{12}$. Since this is larger than π, and the internal angles of a triangle add up to π^c, we know that $\frac{5\pi}{6}$ is not a valid solution.

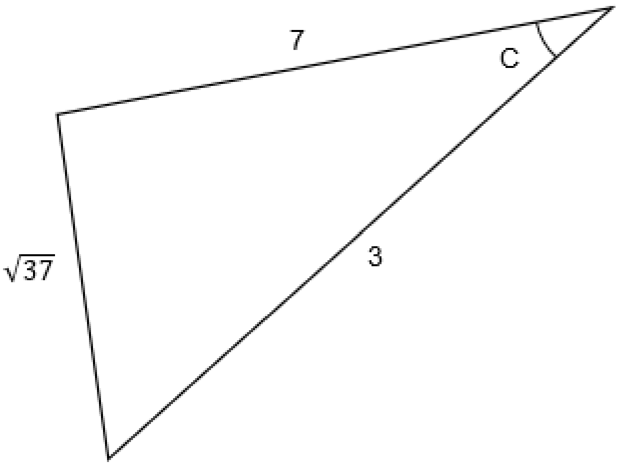
4. $sin(\theta) = \frac{9\,sin\left(\frac{\pi}{30}\right)}{2.5}$, so therefore $\theta = sin^{-1}\left(\frac{9\,sin\left(\frac{\pi}{30}\right)}{2.5}\right) = 0.386^c$. If there is another solution, it will be found by $\pi - 0.386 = 2.756^c$. We can test this by adding the known internal angles: $2.756 + \frac{\pi}{30} = 2.861^c$. Since this is less than π, 2.756^c is a potential answer, and so the angle can be 0.386^c or 2.756^c.

Cosine Rule

Study Development Worksheet

Example

Find the size of angle C:



Answer

Using the cosine rule, we have $a^2 = b^2 + c^2 - 2bc\,cos(A)$.

We fill in the information we have: $\left(\sqrt{37}\right)^2 = 7^2 + 3^2 - 2 \times 7 \times 3 \times cos(A)$.

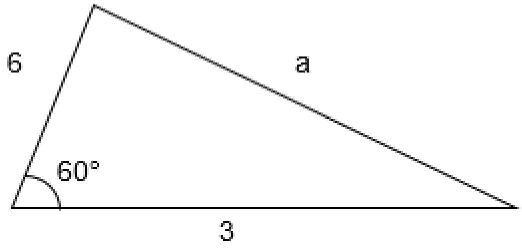
We rearrange and simplify, to get $cos(A) = \frac{49+9-37}{42}$, and so $A = cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

Cosine Rule

Study Development Worksheet

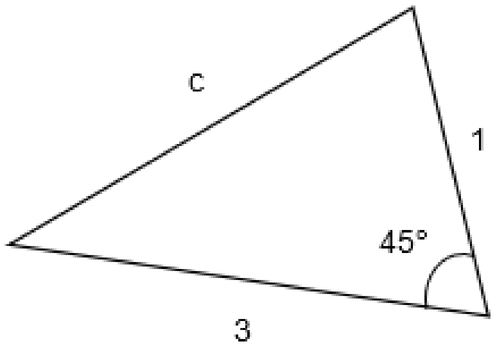
Questions

1. Find the side length a:

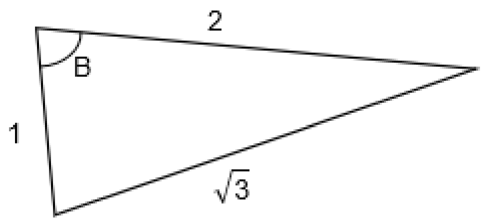


What kind of triangle is this?

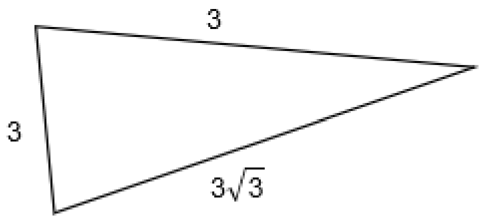
2. Find the side length c:



3. Find the size of angle B in degrees:



4. Find the size of each internal angle in degrees:



Cosine Rule

Study Development Worksheet

Answers

1. We find a using the cosine rule: $a^2 = b^2 + c^2 - 2bc\,cos(A)$. Filling in the information that we are given, we have $a^2 = 3^2 + 6^2 - 2 \times 3 \times 6 \times cos(60)$, so $a^2 = 9 + 36 - \frac{36}{2} = 27$, and therefore $a = \sqrt{27} = 3\sqrt{3}$.

   To tackle the issue of defining the type of triangle, we can check for some of the more common defining features of triangles. We know that the side lengths are 3, 6, and $3\sqrt{3}$. Since these lengths are distinct (ie not the same as each other) we know this cannot be an equilateral or isosceles triangle. We may check if they follow Pythagoras's theorem by adding the squares of the two shorter sides and seeing if this is equal to the square of the longer side:

   $3^2 + \left(3\sqrt{3}\right)^2 = 9 + 27 = 36 = 6^2$. Therefore, the three side lengths follow Pythagoras's theorem. We perform a final check on the angle opposite the longest side length before defining the triangle:

   $$c^2 = a^2 + b^2 - 2ab\,cos(C)$$

   $$6^2 = \left(3\sqrt{3}\right)^2 + 3^2 - 2 \times 3\sqrt{3} \times 3 \times cos(C)$$

   Rearranging, we get

   $$cos(C) = \frac{27 + 9 - 36}{18\sqrt{3}} = 0$$

   Therefore, C = 90°, and we define the triangle as a right-angle triangle. We can also say that this triangle is a scalene triangle since all three sides are different lengths.

2. $c^2 = a^2 + b^2 - 2ab\,cos(C)$

   Therefore, in this question, $c^2 = 1^2 + 3^2 - 2 \times 1 \times 3 \times cos(45) = 10 - 3\sqrt{2}$.

   So, the length of c is $\sqrt{10 - 3\sqrt{2}}$.

Cosine Rule

Study Development Worksheet

3. We rearrange the cosine rule to give us $cos(A) = \frac{b^2+c^2-a^2}{2bc}$.

   In this case, this gives us $cos(A) = \frac{1^2 + 2^2 - (\sqrt{3})^2}{2 \times 1 \times 2} = \frac{1}{2}$. Therefore, we have that $A = cos^{-1}\left(\frac{1}{2}\right) = 60°$.

4. We find the angle opposite $3\sqrt{3}$:

   $cos(A) = \frac{3^2+3^2-(3\sqrt{3})^2}{2\times3\times3} = \frac{-1}{2}$. We find $A = cos^{-1}\left(\frac{-1}{2}\right) = 120°$.

   We may have noticed that we have two side lengths that are equal. By the properties of isosceles triangles, we know that the other two angles must therefore be the same size, so we can find them by B = C = $\frac{180-120}{2}$ = 30°.

   If we were not aware of this property, we could find a second angle in the same way we found the first:

   $cos(B) = \frac{(3\sqrt{3})^2+3^2-3^2}{2\times3\sqrt{3}\times3} = \frac{\sqrt{3}}{2}$. We then find B = $cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ = 30°.

   We can then find the final angle by using the fact that the internal angles in a triangle add up to 180°: 180° - 120° – 30° = 30°.

   Alternatively, we could use the cosine rule again to find the final angle.

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Trigonometry

Study Development Worksheet

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